Integrable field theories with defects

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Classical and Quantum Integrable Models

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Contents

Integrable dynamical systems and field theory have a long history (over 100 years) - with many developments since 1968.

Integrable field theory in the presence of boundaries (one boundary or two), or defects (shocks), is more recent.

The purpose here is to give (from a personal perspective) a small collection of ideas and questions.

- Sine-Gordon field theory - a lightning review
- Bäcklund transformations and defects
- Solitons and defects
- Defects in sine-Gordon quantum field theory

Apology: references are not comprehensive.

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The sine-Gordon field theory

From a physicist's perspective - began with Skyrme (1959-62).

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -\frac{m^2}{\beta}\sin\beta u.$$

- c is a constant with the dimensions of velocity (usually set to unity),
- m is a constant with dimensions of inverse length ($\hbar m$ has the dimensions of mass);
- β sets the scale of the field u: as $\beta \to 0$, s-G $\to$ linear.

All these constants can be removed by scaling t, x and u, though β in particular is important for quantization.

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For the following reasons the sine-Gordon nonlinear wave equation provides a paradigm:

- it is (almost) the simplest (a single scalar field), relativistic, integrable nonlinear wave equation in two dimensions (one time, one space) (t, x);
- it is simple enough to allow direct computations in the classical or quantum domains;
- it is complicated enough to display a wide range of interesting phenomena;
- though originally studied on the range $-\infty < x < \infty$, or on a circle (periodic boundary conditions), there are new features when the model is restricted to a half-line ($x < 0$, say), or to an interval ($x \in [-L, L]$), by suitable boundary conditions, or if there are 'impurities' or 'defects'.

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Expanding the right hand side of the sine-Gordon equation reveals....

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -m^2 u +$$

$$+ \frac{m^2\beta^2}{3!}u^3 - \frac{m^2\beta^4}{5!}u^5 + \dots$$

The first three (linear) terms taken alone are simply the Klein-Gordon equation for a relativistic scalar particle with mass parameter m.

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Energy and momentum

The sine-Gordon equation provides the stationary points of an action given by the Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\partial_\mu u\,\partial^\mu u - \frac{m^2}{\beta^2}(1 - \cos\beta u).$$

The corresponding conserved energy and momentum are given by

$$\mathcal{E} = \int_{-\infty}^{\infty} dx\,\left(\frac{1}{2}(u_t^2 + u_x^2) + \frac{m^2}{\beta^2}(1 - \cos\beta u)\right),$$

$$\mathcal{P} = -\int_{-\infty}^{\infty} dx\,u_t u_x.$$

Well-defined provided u is 'smooth' with $u_t, u_x \to 0$, $\beta u \to 2n\pi$, as $x \to \pm\infty$, where n is an integer or zero.

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Solitons

It is easy to check that the following gives an exact (real) solution to the sine-Gordon equation:

$$e^{i\beta u/2} = \frac{1 + iE}{1 - iE}, \quad E = e^{ax+bt+c},$$

where a, b are real constants satisfying

$$a^2 - b^2 = m^2,$$

and c is a constant that need not be real, but e^c is real.
Note:

- Useful to put $a = m\cosh\theta, \ b = -m\sinh\theta$; and θ is the 'rapidity'.
- We take $a > 0$.

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Properties

Assume first $E > 0$ (ie $e^c > 0$).

- The spatial derivative u_x is given by

$$u_x = \frac{4a}{\beta} \frac{E}{1 + E^2},$$

  which implies u is monotonically increasing.
- As $x \to -\infty$, $e^{i\beta u/2} \to 1$; thus $u \to 0$ is a suitable choice for $x \to -\infty$.
- As $x \to +\infty$, $e^{i\beta u/2} \to -1$; since u is always increasing we must have $u \to 2\pi/\beta$ for $x \to +\infty$.

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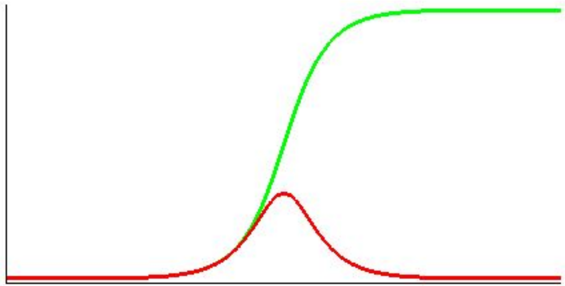
A soliton snapshot



The lower curve represents u_x (and is similar in general shape to the energy density) and the upper curve represents the soliton itself smoothly interpolating $u = 0$ to $u = 2\pi$.

The solution is changing rapidly within a small region in the neighbourhood of $x = 0$.

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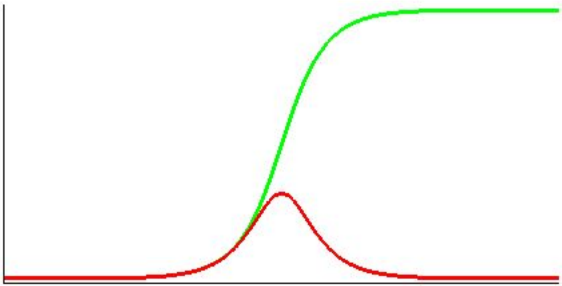

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- For $\theta > 0$ the soliton is travelling along the x-axis in a positive direction with velocity $b/a = \tanh\theta$.
- Its energy and momentum are calculated directly to be

$$(\mathcal{E}, \mathcal{P}) = \frac{8m}{\beta^2}(\cosh\theta, \sinh\theta).$$

  This expression is the energy-momentum of a relativistic particle ($c = 1$) of mass $M = 8m/\beta^2$.
- Note: assigning the units of action (ML) to the action requires $[u]^2 = ML$ and hence $[\beta^2] = 1/ML$ (which is why a physicist might prefer not to put $\beta = 1$). Since $[m] = 1/L$, this means that M has the same dimensions as $\hbar m$, and it corresponds to a classically generated mass.
- A strongly localised field configuration $\sim$ a particle.

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An anti-soliton

Return to the expression for a soliton:

$$e^{i\beta u/2} = \frac{1+iE}{1-iE}, \quad E = e^{ax+bt+c}$$

and replace c by $c + i\pi$ (equivalently, replace E by $-E$). Note

$$u_x = -\frac{4a}{\beta}\frac{E}{1+E^2},$$

which is always negative - this time the solution interpolates from 0 to -2π, with identical energy-momentum.
Define a conserved ('topological') charge

$$Q = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, u_x = \frac{1}{2\pi}[u(t,\infty) - u(t,-\infty)].$$

Then $Q = 1$ for a soliton and $Q = -1$ for an anti-soliton.

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Multi-solitons

It is also possible to check directly (use Maple/Mathematica) that the following expression is also a solution and describes two solitons (stems from the 60s - see any soliton book):

$$e^{i\beta u/2} = \frac{1 + iE_1 + iE_2 - \Omega_{12} E_1 E_2}{1 - iE_1 - iE_2 - \Omega_{12} E_1 E_2}, \quad \Omega_{12} = \tanh^2\left(\frac{\theta_1 - \theta_2}{2}\right),$$

where

$$E_k = e^{a_k x + b_k t + c_k}, \; a_k = m\cosh\theta_k, b_k = -m\sinh\theta_k, \quad k = 1, 2$$

Also

$$(\mathcal{E}, \mathcal{P}) = (\mathcal{E}_1, \mathcal{P}_1) + (\mathcal{E}_2, \mathcal{P}_2),$$

the sum of the individual soliton energies and momenta.

Generalises to any number of solitons (point to note, rapidities are all different).

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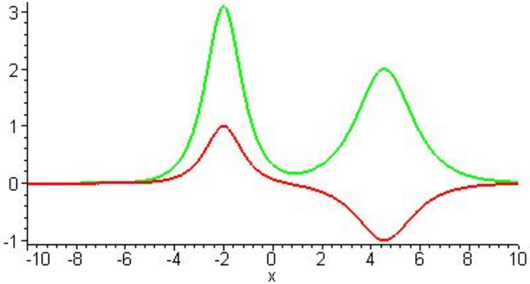
Again, u_x is positive and, taking as example $\theta_1 = 0,\ \theta_2 = 0.5$, two maxima are clearly seen in the regions where the solution is changing rapidly:



In this snapshot the moving soliton is to the left of the stationary one (and the red curve represents $\sin(u/2)$). Since the derivative is always positive, u increases from $0 \rightarrow 4\pi$.

Remarks:

- Either E_1 or E_2 or both can be replaced by $-E_1,\ -E_2$, respectively, to give solutions with soliton-anti-soliton, or two solitons.
- A simple time-periodic solution (known as a 'breather') may be constructed by setting

$$\theta_1 = i\lambda, \ \ \theta_2 = -i\lambda, \ \ c_1 = c_2.$$

- The energy-momentum of this breather is given by

$$(\mathcal{E}, \mathcal{P}) = \frac{16m}{\beta^2}(\cos\lambda, 0) \equiv 2M(\cos\lambda, 0).$$

  Evidently, the energy of a breather is less than the mass of two solitons, indicating a bound-state - further evidence for Skyrme that this was an interesting model to analyse.

Further remarks

- A 'real' version of sine-Gordon is sinh-Gordon $\partial^2 u = -\sinh u$; it is at first sight less interesting because it has no real solitons.
- It is sometimes convenient to use light-cone variables $z = t + x, \bar{z} = t - x$. Then the sinh-Gordon equation reads $4\partial\bar{\partial}u = -\sinh u$.
- The Liouville equation is simpler-looking: $4\partial\bar{\partial}u = -e^u$. It is also conformally invariant under the transformation

$$z \to z'(z),\ \bar{z} \to \bar{z}'(\bar{z}),\ u' = u + \ln\left(\frac{d\bar{z}'}{d\bar{z}}\frac{dz'}{dz}\right)$$

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Affine Toda field theory

The sinh/sine-Gordon model is the simplest of a large class of field theories based on Lie algebra data (the sinh/sine-Gordon model is based on the roots of a_1 or su(2)).

In many respects the whole class may be considered together - though the sinh/sine-Gordon model is particularly special - they are all integrable in a sense that generalises Liouville's theorem for finite dynamical systems (meaning there are 'enough' conserved quantities in involution).

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Bäcklund transformations

Return for a while to the sine-Gordon equation we began with

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -\frac{m^2}{\beta}\sin\beta u,$$

or, alternatively, scaling away all constants, $u_{tt} - u_{xx} = -\sin u$.

A remarkable observation of Bäcklund (1882) concerns two solutions to the sine-Gordon equation related by first order differential equations:

$$\begin{aligned} u_x &= v_t + \lambda \sin\left(\frac{u+v}{2}\right) + \lambda^{-1}\sin\left(\frac{u-v}{2}\right) \\ v_x &= u_t - \lambda \sin\left(\frac{u+v}{2}\right) + \lambda^{-1}\sin\left(\frac{u-v}{2}\right). \end{aligned}$$

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Eliminating v gives the sine-Gordon equation for u, and vice-versa.

The first interesting remark concerns the choice $v = 0$. With this choice u satisfies:

$$\begin{aligned} u_x &= \left(\lambda + \lambda^{-1}\right) \sin\left(\frac{u}{2}\right) \\ u_t &= \left(\lambda - \lambda^{-1}\right) \sin\left(\frac{u}{2}\right), \end{aligned}$$

whose solution is precisely the single soliton we had at the beginning provided we identify $\lambda = e^{\theta}$, where θ is the soliton's rapidity.

That is, u is given by

$$e^{iu/2} = \frac{1 + iE}{1 - iE}, \quad E = e^{ax+bt+c},$$

with $a = \cosh\theta,\ b = -\sinh\theta$.

The second point concerns energy and momentum, which are each clearly seen to be boundary terms. For example:

$$\mathcal{P} = -\int_{-\infty}^{\infty} dx\, u_t u_x = -\int_{-\infty}^{\infty} dx \left(\lambda - \lambda^{-1}\right) \sin\left(\frac{u}{2}\right)\, u_x.$$

Hence,

$$\mathcal{P} = \left(\lambda - \lambda^{-1}\right) \left[\cos\left(\frac{u}{2}\right)\right]_{-\infty}^{\infty} = -4\sinh\theta.$$

A similar argument yields the energy as a boundary contribtion

$$\mathcal{E} = -\left(\lambda + \lambda^{-1}\right) \left[\cos\left(\frac{u}{2}\right)\right]_{-\infty}^{\infty} = 4\cosh\theta.$$

It was also noted that the Bäcklund transformation can be used to generate multiple solitons. For example, taking v be a single soliton and solving for u leads to a double-soliton.

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An almost physical example - a defect or shock

Bowcock, EC, Zambon (2002)

Typical shock (or bore) in fluid mechanics:

- flow flips from supersonic to subsonic,

- abrupt change of depth in a channel.

- Velocity field changes rapidly over a small distance,
- Model by a discontinuity in $\mathbf{v}(\mathbf{x}, t)$,
- There remain conserved quantities - mass, momentum, for example.

- Are shocks allowed in integrable QFT?
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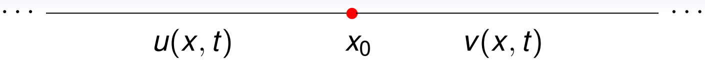

Start with a single selected point on the x-axis, say $x = 0$, and denote the field to the left of it ($x < 0$) by u, and to the right ($x > 0$) by v, with field equations in their respective domains:

$$\partial^2 u = -\frac{\partial U}{\partial u}, \quad x < 0, \quad \partial^2 v = -\frac{\partial V}{\partial v}, \quad x > 0$$

- How can the fields be 'sewn' together preserving integrability?

One natural choice (δ-impurity) would be to put

$$u(0,t) = v(0,t), \quad u_x(0,t) - v_x(0,t) = \mu\, u(0,t),$$

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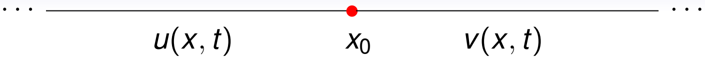

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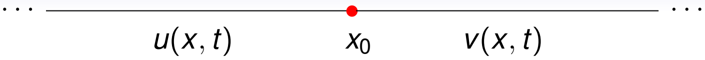

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• Problem: there is a distinguished point, translation symmetry is lost and the conservation laws - at least some of them - (for example, momentum), are violated unless the impurity has the property of adding by itself compensating terms.

Consider the field contributions to momentum:

$$\mathcal{P} = -\int_{-\infty}^{0} dx\, u_t u_x - \int_{-\infty}^{0} dx\, v_t v_x .$$

Then, using the field equations, $2\dot{\mathcal{P}}$ is given by

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If there are ‘sewing’ conditions for which the last step is valid then $\mathcal{P} + \Omega$ will be conserved, with Ω a function of u, v - and possibly derivatives - evaluated at $x = 0$.

Next, consider the energy density and calculate

$$\dot{\mathcal{E}} = [u_x u_t]_0 - [v_x v_t]_0.$$

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This is a total time derivative provided

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If there are 'sewing' conditions for which the last step is valid then $\mathcal{P} + \Omega$ will be conserved, with Ω a function of u, v - and possibly derivatives - evaluated at $x = 0$.

Next, consider the energy density and calculate

$$\dot{\mathcal{E}} = [u_x u_t]_0 - [v_x v_t]_0.$$

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This is a total time derivative provided the first piece is a perfect differential and the second piece vanishes. Thus....

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Exercise Investigate the possible combinations U, V, D.

... should find those allowed are: sine-Gordon, Liouville, massless free, or, massive free.

For example, $U(u) = m^2u^2/2$, $V(v) = m^2v^2/2$, D turns out to be

$$D(u, v) = \frac{m\sigma}{4}(u + v)^2 + \frac{m}{4\sigma}(u - v)^2,$$

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It is also worth noting there is a Lagrangian description of this type of 'shock':

$$\mathcal{L} \;=\; \theta(-x)\mathcal{L}(u) + \delta(x)\left(\frac{uv_t - u_t v}{2} - D(u,v)\right) + \theta(x)\mathcal{L}(v)$$

The usual E-L equations provide both the field equations for u, v in their respective domains **and** the 'sewing' conditions.

Exercise in the free case, what happens to a wave incident from (say) the left half-line?

Show that if

$$u = \left(e^{ikx} + Re^{-ikx}\right)e^{-i\omega t}, \quad v = Te^{ikx}e^{-i\omega t}, \quad \omega^2 = k^2 + m^2,$$

then $R = 0$ and find T. (At first sight this seems surprising.)

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sine-Gordon

Choosing u, v to be sine-Gordon fields (and scaling the coupling and mass parameters to unity), we take:

$$D(u,v) = 2\left(\sigma\cos\frac{u+v}{2} + \sigma^{-1}\cos\frac{u-v}{2}\right)$$

to find

$$\begin{aligned}
x < x_0: \quad \partial^2 u &= -\sin u,\\
x > x_0: \quad \partial^2 v &= -\sin v,\\
x = x_0: \quad u_x &= v_t - \sigma\sin\frac{u+v}{2} - \sigma^{-1}\sin\frac{u-v}{2},\\
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The last two expressions are a Bäcklund transformation frozen at $x = x_0$.

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Solitons and defects

Consider a soliton incident from $x < 0$.

It will not be possible to satisfy the sewing conditions (in general) unless a similar soliton emerges into the region $x > 0$:

$$e^{iu/2} = \frac{1 + iE}{1 - iE}, \quad e^{iv/2} = \frac{1 + izE}{1 - izE}, \quad E = e^{ax+bt+c},$$
$$a = \cosh\theta, \quad b = -\sinh\theta,$$

where z is to be determined. It is also useful to set $\lambda = e^{-\eta}$.

- We find

$$z = \coth\left(\frac{\eta - \theta}{2}\right).$$

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- $\eta < \theta$ implies $z < 0$; ie the soliton emerges as an anti-soliton.

-The final state will contain a discontinuity of magnitude 4π at $x = 0$.

- $\eta = \theta$ implies $z = \infty$ and there is **no** emerging soliton.

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Comments and questions....

• The shock is local so there could be several shocks located at $x = x_1 < x_2 < x_3 < \cdots < x_n$; these behave independently as far as a soliton is concerned, each contributing a factor z_i for a total 'delay' of $z = z_1 z_2 \ldots z_n$.

• When several solitons pass a defect each component is affected separately.

- This means that at most one of them can be 'filtered out' (since the components of a multisoliton in the sine-Gordon model must have different rapidities).

• Since a soliton can be absorbed, can a starting configuration with $u = 0$, $v = 2\pi$ decay into a soliton?

- No, there is no way to tell the time at which the decay would occur (and quantum mechanics would be needed to provide the probability of decay as a function of time).

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The classical type II defect

Consider two relativistic field theories with fields u and v, and add a new degree of freedom $\lambda(t)$ at the defect location:

$$\mathcal{L} = \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v + \delta(x)\left(2q\lambda_t - D(\lambda, p, q)\right)$$

where

$$q = \left.\frac{u-v}{2}\right|_0 \qquad p = \left.\frac{u+v}{2}\right|_0 .$$

Then the usual steps lead to

- equations of motion:

$$\partial^2 u = -U_u \quad x < 0 \qquad \partial^2 v = -V_v \quad x > 0$$

- defect conditions at $x = 0$

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As before, consider momentum

$$P = \int_{-\infty}^{0} dx\, u_t u_x + \int_{0}^{\infty} dx\, v_t v_x,$$

and seek a functional $\Omega(u, v, \lambda)$ such that $P_t \equiv -\Omega_t$. Then $P + \Omega|_{x=0}$ is the total conserved momentum of the system.

Constraints on U, V, Ω:

$$D_p = \Omega_\lambda \qquad D_\lambda = \Omega_p \qquad D_p D_q - \Omega_q D_\lambda = 2(U - V),$$

implying

$$D = f(p+\lambda, q) + g(p-\lambda, q) \qquad \Omega = f(p+\lambda, q) - g(p-\lambda, q)$$

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- Exercise — show that it is now possible to choose f, g in such a way that the potentials U, V can be any one of sine-Gordon, Liouville, Tzitzéica, or quadratic. Are there solutions other than the integrable cases?
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Shocks in sine-Gordon quantum field theory

Assume $\sigma > 0$ then...

- Expect Pure transmission compatible with the bulk S-matrix;
- Expect For each type of defect two different 'transmission' matrices (since the topological charge on a defect can only change by ± 2 as a soliton/anti-soliton passes).
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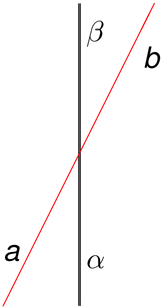

Schematic triangle relation Delfino, Mussardo, Simonetti 1994

$$S^{cd}_{ab}(\Theta)\, T^{f\beta}_{d\alpha}(\theta_a) T^{e\gamma}_{c\beta}(\theta_b) = T^{d\beta}_{b\alpha}(\theta_b) T^{c\gamma}_{a\beta}(\theta_a) S^{ef}_{cd}(\Theta)$$

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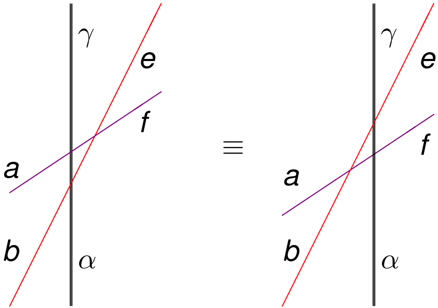

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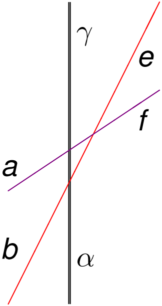

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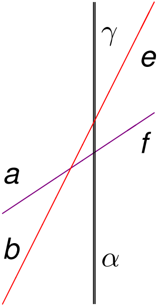

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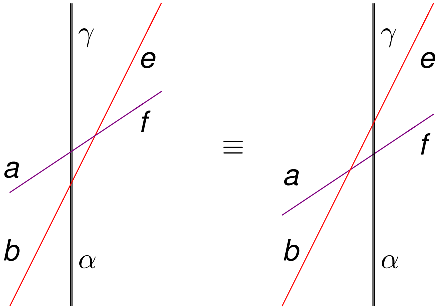

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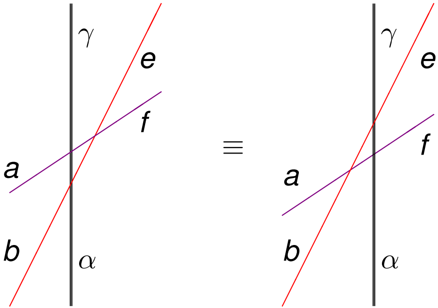

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Zamolodchikov's sine-Gordon S-matrix - reminder

$$S_{ab}^{cd}(\Theta) = \rho(\Theta) \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & C & B & 0 \\ 0 & B & C & 0 \\ 0 & 0 & 0 & A \end{pmatrix}$$

where

$$A(\Theta) = \frac{qx_2}{x_1} - \frac{x_1}{qx_2},\ B(\Theta) = \frac{x_1}{x_2} - \frac{x_2}{x_1},\ C(\Theta) = q - \frac{1}{q}$$

and

$$\begin{aligned} \rho(\Theta) &= \frac{\Gamma(1+z)\Gamma(1-\gamma-z)}{2\pi i} \prod_1^\infty R_k(\Theta) R_k(i\pi - \Theta) \\ R_k(\Theta) &= \frac{\Gamma(2k\gamma + z)\Gamma(1 + 2k\gamma + z)}{\Gamma((2k+1)\gamma + z)\Gamma(1 + (2k+1)\gamma + z)},\ z = i\gamma/\pi. \end{aligned}$$

The Zamolodchikov S-matrix depends on the rapidity variables θ and the bulk coupling β via

$$x = e^{\gamma\theta},\ q = e^{i\pi\gamma},\ \gamma = \frac{8\pi}{\beta^2} - 1,$$

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$$Q = e^{4\pi^2 i/\beta^2} = \sqrt{-q}.$$

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Remarks ($\theta > 0$): it is tempting to suppose η (possibly renormalized) is the same parameter as in the classical model.

- $\eta < 0$ - the off-diagonal entries dominate;
- $\theta > \eta > 0$ - the off-diagonal entries dominate;
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• These are the same features we saw in the classical soliton-shock scattering.

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- This pole is like a resonance, with complex energy,

$$E = m_s \cosh\theta = m_s(\cosh\eta\cos(\pi/2\gamma) - i\sinh\eta\sin(\pi/2\gamma))$$

and a 'width' proportional to $\sin(\pi/2\gamma)$.

Using this pole and a bootstrap to define ${}^{odd}T$ leads to a non-unitary transmission matrix - interpret as the instability corresponding to the classical feature noted at $\theta = \eta$.

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use the bootstrap to define the transmission factors for breathers and find for the lightest breather:

$$T(\theta) = -i\frac{\sinh\left(\frac{\theta-\eta}{2} - \frac{i\pi}{4}\right)}{\sinh\left(\frac{\theta-\eta}{2} + \frac{i\pi}{4}\right)}$$

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Further questions....

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- Some alternative views and other aspects are discussed in several places. (For example Habibullin, Kundu, 2007)

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