Exercise related to Corrigan's lecture

Start from the sine-Gordon equation normalised as

$$u_{tt} - u_{xx} = 4 \sin u . \qquad (1)$$

Let $e^{iu/2} = \frac{\tau_+}{\tau_-}$, for some functions $\tau_\pm$ ("tau-functions")

Ex 1 Verify that if $\tau_\pm$ satisfy the two equations

$$\tau_\pm \frac{\partial^2 \tau_\pm}{\partial t^2} - \left(\frac{\partial \tau_\pm}{\partial t}\right)^2 - \tau_\pm \frac{\partial^2 \tau_\pm}{\partial x^2} + \left(\frac{\partial \tau_\pm}{\partial x}\right)^2 = \tau_\pm^2 - \tau_\mp^2 ,$$

then u satisfies (1).

(Note: these equations for $\tau_\pm$ are "bilinear", in the sense that both dependent variables appear quadratically. Defining Hirota derivatives

$$D_x^j D_t^k \, \tau_1 \circ \tau_2 \,(x,t) := \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^j \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k \tau_1(x,t)\tau_2(x',t') \Big|_{\substack{x=x' \\ t=t'}}$$

the bilinear equations take the concise form

$$(D_t^2 - D_x^2)\, \tau_\pm \circ \tau_\pm = 2(\tau_+^2 - \tau_-^2) . \;)$$

Ex. 2 Verify that the bilinear equations for $\tau_\pm$ have a solution given by

$$\tau_\pm = 1 \pm e^{ax + bt + c}$$

for suitable a, b, c satisfying a constraint. Hence reproduce the 1-soliton solution as in Corrigan's lecture (up to rescaling).

[Note: $\tau_- = \overline{\tau}_+$ which ensures reality of u.]

Ex. 3 Use the bilinear equations to verify the formula for the 2-soliton as in the lecture.

[Hint: $D_x^j D_t^k \, e^{a_1 x + b_1 t + c_1} \circ e^{a_2 x + b_2 t + c_2} = (a_1 - a_2)^j (b_1 - b_2)^k e^{(a_1+a_2)x + (b_1+b_2)t + c_1 + c_2}$]