EXAMPLE CLASS 2

(HAMILTONIAN APPROACH TO INTEGRABLE DISCRETIZATION)

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1. Main properties of the Toda Lattice

▷ Equations of motion of the (finite-dimensional) Toda Lattice (TL) read:

$$\ddot{x}_k = e^{x_{k+1}-x_k} - e^{x_k - x_{k-1}}, \qquad 1 \le k \le N,$$

with $x_0 = \infty, x_{N+1} = -\infty$ (open-end boundary conditions) or $x_0 = x_N, x_{N+1} = x_1$ (periodic boundary conditions).

▷ If one introduces the variables $b_k = \dot{x}_k$, $a_k = e^{x_{k+1}-x_k}$, then TL is governed by

$$\begin{cases} \dot{b}_k = a_k - a_{k-1}, \\ \dot{a}_k = a_k(b_{k+1} - b_k), \end{cases} \tag{1}$$

with $a_0 = a_N = 0$ (open-end boundary conditions) or $a_0 = a_N, b_{N+1} = b_1$ (periodic boundary conditions).

▷ The phase-space of TL is

$$\mathcal{T} := \mathbb{R}^{2N}(a_1, ..., a_N, b_1, ..., b_N),$$

in the periodic case, and

$$\mathcal{T}_0 := \mathbb{R}^{2N-1}(a_1, ..., a_{N-1}, b_1, ..., b_N) = \{(a, b) \in \mathcal{T} : a_N = 0\},$$

in the open-end case.

▷ TL is tri-Hamiltonian: there exist three Hamiltonian formulations w.r.t. three independent but compatible Poisson brackets. In particular, the first two Hamiltonian formulations are:

(1) TL is Hamiltonian on the Poisson manifold $(\mathcal{T}, \{\cdot,\cdot\}_1)$ (on $(\mathcal{T}_0, \{\cdot,\cdot\}_1)$ in the open-end case) with the Hamiltonian function

$$H = \frac{1}{2}\sum_{k=1}^{N} b_k^2 + \sum_{k=1}^{N} a_k,$$

where the Poisson bracket $\{\cdot,\cdot\}_1$ is defined by

$$\{b_k, a_k\}_1 = -a_k, \qquad \{a_k, b_{k+1}\}_1 = -a_k. \tag{2}$$

(2) TL is Hamiltonian on the Poisson manifold $(\mathcal{T}, \{\cdot,\cdot\}_2)$ (on $(\mathcal{T}_0, \{\cdot,\cdot\}_2)$ in the open-end case) with the Hamiltonian function

$$H = \sum_{k=1}^{N} b_k,$$

where the Poisson bracket $\{\cdot,\cdot\}_2$ is defined by

$$\{b_k, a_k\}_2 = -b_k a_k, \qquad \{a_k, b_{k+1}\}_2 = -a_k b_{k+1}, \tag{3}$$

$$\{b_k, b_{k+1}\}_2 = -a_k, \qquad \{a_k, a_{k+1}\}_2 = -a_k a_{k+1}. \tag{4}$$

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Exercise 1. *Prove that the functions*

$$C_1 = \prod_{k=1}^{N} a_k, \qquad C_2 = \sum_{k=1}^{N} b_k,$$

are two polynomial Casimir functions of $(\mathcal{T}, \{\cdot,\cdot\}_1)$.

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▷ TL has several Lax representations and it admits an r-matrix structure. A Lax representation of TL is:

$$\dot{L}(\lambda) = [L(\lambda), B(\lambda)] = -[L(\lambda), A(\lambda)], \qquad \lambda \in \mathbb{C},$$

with

$$L(\lambda) = \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1} + \sum_{k=1}^{N} b_k E_{k,k} + \lambda \sum_{k=1}^{N} E_{k+1,k},$$

$$B(\lambda) = \sum_{k=1}^{N} b_k E_{k,k} + \lambda \sum_{k=1}^{N} E_{k+1,k},$$

$$A(\lambda) = \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1},$$

where the matrices $E_{j,k}$, $(E_{i,j})_{k,\ell} = \delta_{ik}\delta_{j\ell}$, form a basis of $\mathfrak{gl}(N)$. We set $E_{N+1,N} = E_{1,N}$, $E_{N,N+1} = E_{N,1}$ in the periodic case and $E_{N+1,N} = E_{N,N+1} = 0$ and $\lambda = 1$ in the open-end case.

Spectral invariants of the Lax matrix $L(\lambda)$ serve as integrals of motion of TL. All spectral invariants are in involution w.r.t. the brackets (2) and (3,4).

▷ As every multi-Hamiltonian and completely integrable system, TL belongs to a whole *integrable hierarchy*. Each flow of this hierarchy may be solved by means of a matrix factorization (finite-dimensional analog of the inverse scattering method). This is the ground for TL do deliver a transparent model for the integrable discretization problem. The resulting discretization shares invariant Poisson structures and integrals of motion with TL. In short, it belongs to the hierarchy attached to TL.

2. An integrable discretization of the Toda Lattice

▷ The discrete Toda lattice (dTL) is governed by the following discrete Lax equation:

$$\mathbb{1} + \epsilon\widetilde{L}(\lambda) = B^{-1}(\lambda;\epsilon)(\mathbb{1} + \epsilon L(\lambda))B(\lambda;\epsilon) = A(\lambda;\epsilon)(\mathbb{1} + \epsilon L(\lambda))A^{-1}(\lambda;\epsilon), \tag{5}$$

where

$$B(\lambda;\epsilon) = \sum_{k=1}^{N} \beta_k E_{k,k} + \epsilon\lambda \sum_{k=1}^{N} E_{k+1,k},$$
$$A(\lambda;\epsilon) = \mathbb{1} + \epsilon\lambda^{-1} \sum_{k=1}^{N} \alpha_k E_{k,k+1}.$$

- Tilde denotes the shift $t \mapsto t + \epsilon$ in the discrete time $\epsilon\mathbb{Z}$; ϵ is a time step.
- The coefficients β_k and α_k are uniquely defined, for ϵ small enough, by the system

$$\begin{cases} \beta_k + \epsilon^2\alpha_{k-1} = 1 + \epsilon b_k, \\ \beta_k\alpha_k = a_k. \end{cases} \tag{6}$$

We have: $\beta_k = 1 + \epsilon b_k + O(\epsilon^2)$.

Remark 1. *In the open-end case system (6) is uniquely solvable not only for small ϵ. Indeed, one obtains explicit expressions in terms of continued fractions:*

$$\beta_1 = 1 + \epsilon b_1,$$
$$\beta_2 = 1 + \epsilon b_2 - \frac{\epsilon^2 a_1}{1 + \epsilon b_1},$$
$$\dots$$
$$\beta_N = 1 + \epsilon b_N - \cfrac{\epsilon^2 a_{N-1}}{1 + \epsilon b_{N-1} - \cfrac{\epsilon^2 a_{N-2}}{1 + \epsilon b_{N-2} - \ddots - \cfrac{\epsilon^2 a_1}{1 + \epsilon b_1}}}.$$

Remark 2. *The Lax equation (5) is equivalent the following map $(a, b) \mapsto (\widetilde{a}, \widetilde{b})$:*

$$\begin{cases} \widetilde{b}_k = b_k + \epsilon\left(\dfrac{a_k}{\beta_k} - \dfrac{a_{k-1}}{\beta_{k-1}}\right), \\ \widetilde{a}_k = a_k \dfrac{\beta_{k+1}}{\beta_k}. \end{cases} \tag{7}$$

3. Localizing changes of variables

▷ dTL (7), when compared with its continuous limit TL (1), has one unpleasent property: equations are *non-local* because of the coefficients β_k and α_k. The functions β_j depend (implicitly) on all a_j, b_j in the periodic case and (explicitly) on all a_j with $j < k$ and all b_j with $j \le k$ in the open-end case.

▷ A way to solve this drawback is the notion of *localizing changes of variables.*

- Consider a lattice system with local interactions between neighboring variables:
$$\dot{x}_k = f_{k \bmod m}(x) = f(x_k, x_{k\pm 1}, ..., x_{k\pm s}). \tag{8}$$
Here $s \in \mathbb{N}$ is the *locality radius* and $m \in \mathbb{N}$ is the *number of fields* of the lattice system. Note that TL (1) has $(s, m) = (1, 2)$.

- Suppose to have integrable difference equations (discretizing (8)) of the form
$$\widetilde{x}_k = x_k + \epsilon \Phi_k(x; \epsilon), \tag{9}$$
where Φ_k depends on all x_j.

- **Problem**: To find changes of variables $X \mapsto x$ which are close to the identity and such that equations (9) take the form
$$\widetilde{X}_k = X_k + \epsilon \Psi_{k \bmod m}(X, \widetilde{X}; \epsilon), \qquad \Psi_{k \bmod m}(X, \widetilde{X}; 0) = f_{k \bmod m}(x), \tag{10}$$
where Ψ_k depends only on $X_j, \widetilde{X}_j$ with $|j - k| \le s$. Such implicit local equations are much better suited for the purposes of numerical simulation.

- It is by no means evident that such localizing variables exist. They are usually defined by the formulas
$$x_k = X_k + \epsilon F_{k \bmod m}(X; \epsilon), \tag{11}$$
with local functions F_k depending only on X_j with $|j-k| \le s$. The inverse change of variable $x \mapsto X$ is always described by non-local formulas.

Remark 3. *Nothing guarantees* a priori *that the pull-back of (8) under the change of variables (11) will be given by local formulas. Nevertheless it often turns out to be the case. This is a way to produce new one-parameter families of* integrable deformations *(or modifications) of (8). See Exercise 4.*

Nothing guarantees a priori *that pull-backs of local Poisson structures under the change of variables (11) are also given by local formulas. In the multi-Hamiltonian cases it often turns out that pull-backs of certain* linear combinations *of invariant Poisson brackets are local again. See Exercise 3.*

4. Localizing changes of variables for the Toda lattice

$\triangleright$ The localizing change of variables for dTL (7) is given by the map $\mathcal{T}(A, B) \mapsto \mathcal{T}(a, b)$ defined by
$$\begin{cases} b_k = B_k + \epsilon A_{k-1}, \\ a_k = A_k(1 + \epsilon B_k). \end{cases} \tag{12}$$

Remark 4. *Note that formulas (12) coincide with (6) upon the identication*
$$\beta_k = 1 + \epsilon B_k, \qquad \alpha_k = A_k. \tag{13}$$
Thus, in the coordinates (A, B) the functions α_k and β_k acquire local *expressions.*

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Exercise 2. *Prove that the pull-back of dTL (7) under the change of variables (12) is given by the following* local *equations:*

$$\begin{cases} \widetilde{B}_k = B_k + \epsilon(A_k - \widetilde{A}_{k-1}), \\ \widetilde{A}_k(1+\epsilon\widetilde{B}_k) = A_k(1+\epsilon B_{k+1}). \end{cases}$$

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Exercise 3.

(1) *Prove that the change of variables (12) is Poisson w.r.t. the brackets*

$$\{B_k, A_k\} = -A_k(1+\epsilon B_k), \qquad \{A_k, B_{k+1}\} = -A_k(1+\epsilon B_{k+1}), \tag{14}$$

if the space $\mathcal{T}(a,b)$ is equipped with the bracket

$$\{\cdot,\cdot\}_1 + \epsilon\{\cdot,\cdot\}_2,$$

where $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ are defined in (2) and (3,4).

(2) *Consider the pull-backs under the change of variables (12) of the Poisson bracket $\{\cdot,\cdot\}_1$ (instead of the combination $\{\cdot,\cdot\}_1 + \epsilon\{\cdot,\cdot\}_2$). Verify that in this case pull-backs are described by highly non-local and non-polynomials formulas.*

Hint: Calculate the Poisson brackets in the following natural order:

$$\begin{aligned} &\{B_1, A_1\}, \\ &\{B_1, B_2\},\ \{A_1, B_2\}, \\ &\{B_1, A_2\},\ \{A_1, A_2\},\ \{B_2, A_2\}, \\ &\{B_1, B_3\},\ \{A_1, B_3\},\ \{B_2, B_3\},\ \{A_2, B_3\}, \\ &\quad\dots \end{aligned}$$

In the case *(1)* expressions for the pairwise Poisson brackets of the quantities (A, B) quickly stabilize to (14).

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Exercise 4. *Prove that the pull-back of TL (1) under the change of variables (12) gives the following* modified TL*:*

$$\begin{cases} \dot{B}_k = (1+\epsilon B_k)(A_k - A_{k-1}), \\ \dot{A}_k = A_k(B_{k+1} - B_k). \end{cases}$$

Hint: The above equations are Hamiltonian w.r.t. the brackets (14) with Hamiltonian function

$$H = \epsilon^{-1}\sum_{j=1}^{N}(B_j + \epsilon A_{j-1}).$$

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