Twistor actions, grassmannians and correlahedra

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review with Adamo, Bullimore & Skinner 1104.2890, & work with Lipstein arxiv:1212.6228, 1307.1443, more recently with Eden, Heslop, Agarwala.

[Also. work by Alday, Arkani-Hamed, Cachazo, Caron-Huot, Drummond, Heslop, Korchemsky, Maldacena, Sokatchev, Trnka. (Annecy, Oxford, Perimeter and Princeton IAS).]

Twistors, amplitudes, Wilson loops & 'hedra

Background [Penrose,Boels, M, Skinner, Adamo, Bullimore...]:

- Twistor space $\mathbb{CP}^{3|4} \leftrightarrow$ space-time; Klein correspondence.
- $N = 4$ Super Yang-Mills has twistor action in twistor space.
- Axial gauge Feynman diagrams $\rightsquigarrow$ 'MHV diagrams'. for amplitudes, Wilson loops & Correlators.
- Planar Wilson-loop/amplitude duality is planar duality for MHV diagrams.
- Super Amplitude/Correlator/Wilson-loop triality.

Focus of this talk [with Lipstein, Agarwala, Eden & Heslop]:

- Twistor Feynman rules give grassmannian and 'hedra formulations.
- Feynman diagrams define cells in positive grassmannian.
- Potentially tile amplituhedra and other correlahedra.

$\mathcal{N} = 4$ Super Yang-Mills

The harmonic oscillator of the 21st century?

- Toy version of standard model, contains QCD and more classically.
- Best behaved nontrivial 4d field theory (UV finite, superconformal $SU(2,2|4)$ symmetry, ...).
- Particle spectrum

helicity	-1	-1/2	0	1/2	1
# of particles	1	4	6	$\bar{4}$	1

- Susy changes helicity so particles form irrep of 'super'-group $\mathrm{SU}(2,2|4)$ like single particle.
- 'completely integrable' in planar (large N) sector.
- much twistor geometry in their amplitudes:
  1. (Ambi-)Twistor string & action descriptions,
  2. Grassmannian residue formulae,
  3. polyhedra volumes $\rightsquigarrow$ the amplituhedron,

Scattering amplitudes for $\mathcal{N} = 4$ super Yang-Mills

4-Momentum:

$$p = (E, p_1, p_2, p_3) = E(1, v_1, v_2, v_3),$$

massless$\leftrightarrow |\mathbf{v}| = c = 1 \Leftrightarrow p \cdot p := E^2 - p_1^2 - p_2^2 - p_3^2 = 0. \Leftrightarrow$

$$\Leftrightarrow \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} E + p_3 & p_1 + ip_2 \\ p_1 - ip_2 & E - p_3 \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix} \begin{pmatrix} \tilde\lambda_{0'} & \tilde\lambda_{1'} \end{pmatrix}$$

Supermomentum: $P = (\lambda, \tilde\lambda, \eta) \in \mathbb{C}^{4|0} \times \mathbb{C}^{0|4}$, where $\eta_i, i = 1, \ldots, 4$ anti-commute $\rightsquigarrow$ wave functions:

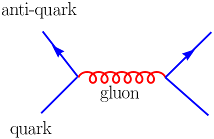
$$\Psi(P) = a_+ + \psi^i \eta_i + \phi^{ij}\eta_i\eta_j + \tilde\psi^{ijk}\eta_i\eta_j\eta_k + a_-\eta_1\eta_2\eta_3\eta_4 .$$

Amplitude: for n-particle process is

$$\mathcal{A}(1, \ldots, n) = \mathcal{A}(P_1, \ldots, P_n)$$

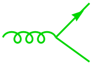
MHV degree: $k \in \mathbb{Z}$, $k + 2 = \#$ of $-$ve helicity particles,
susy $\Rightarrow \mathcal{A} = 0$ for $k = -1, -2$
For $k = 0$, $\mathcal{A} \neq 0$, 'Maximal Helicity Violating' (MHV).

Contributions

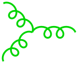


Feynman diagrams are more than pictures. They represent algebraic formulas for the propagation and interaction of particles.

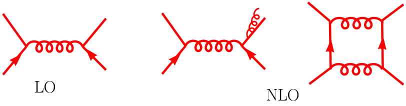
$\frac{\eta_{\mu\nu}}{p^2}$



$g\,\gamma_\mu$



$g[\,(p_1 - p_2)_\rho \eta_{\mu\nu} + (p_2 - p_3)_\mu \eta_{\nu\rho} + (p_3 - p_1)_\nu \eta_{\rho\mu}\,]$



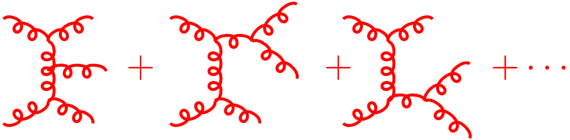
Trees $\leftrightarrow$ classical, loops $\leftrightarrow$ quantum.

Locality: only simple poles from propagators at $(\sum p_i)^2 = 0$.

Need for new ideas

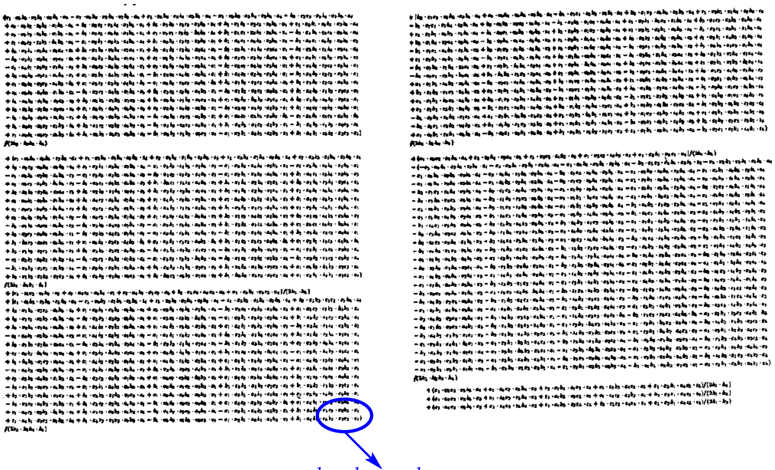
Consider the five-gluon tree-level amplitude of QCD. Enters in calculation of multi-jet production at hadron colliders.

Described by following Feynman diagrams:



If you follow the textbooks you discover a disgusting mess.

Result of a brute force calculation:



$k_1 \cdot k_4\, \varepsilon_2 \cdot k_1\, \varepsilon_1 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5$

However, result for helicity $(+ + - - -)$ part of the amplitude is

$$\mathcal{A}(1,2,3,4,5) = \delta\left(\sum_{a=1}^{5} p_a\right) \frac{\langle\lambda_b\,\lambda_c\rangle^4}{\langle\lambda_1\,\lambda_2\rangle\langle\lambda_2\,\lambda_3\rangle\langle\lambda_3\,\lambda_4\rangle\langle\lambda_4\,\lambda_5\rangle\langle\lambda_5\,\lambda_1\rangle}$$

where b and c are the $+$ helicity particles and

$$\langle i\,j\rangle := \langle\lambda_i\,\lambda_j\rangle := \lambda_{i0}\lambda_{j1} - \lambda_{i1}\lambda_{j0}$$

(similarly use $[i,j]$ for $\tilde{\lambda}_i$s).

More generally (Parke-Taylor 1984, Nair 1986)

$$\mathcal{A}^{tree}_{MHV}(1,\ldots,n) = \frac{\delta^{4|8}\left(\sum_{a=1}^{n}(p_a, \eta_a\lambda_a)\right)}{\prod_{a=1}^{n}\langle\lambda_a\,\lambda_{a+1}\rangle}$$

Super twistor space is $\mathbb{CP}^{3|4}$ with homogeneous coords:

$$Z = (\lambda_\alpha, \mu^{\dot\alpha}, \chi_a) \in \mathbb{T} := \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^{0|4}, \qquad Z \sim \zeta Z\,, \zeta \in \mathbb{C}^*\,.$$

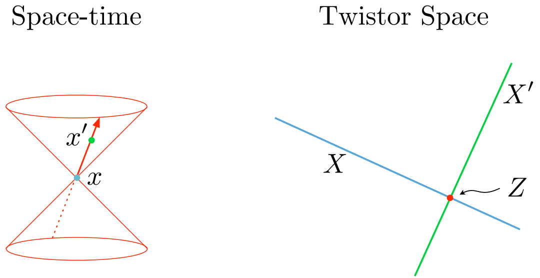
$\mathbb{T}$ = fund. repn of superconformal group $SU(2,2|4)$.

Super Minkowski space, $\mathbb{M} = G(2,4|4) \supset \mathbb{R}^{4|8}$,

Incidence: a point $\mathbf{x} = (x, \theta) \leftrightarrow$ a line $X = \mathbb{CP}^1 \subset \mathbb{PT}$ via

$$\mu^{\dot\alpha} = i x^{\alpha\dot\alpha}\lambda_\alpha\,, \qquad \chi_i = \theta_i^\alpha \lambda_\alpha\,.$$

Two points x, x' are null separated iff X and X' intersect.



Supersymmetric Ward correspondence

Super Calabi-Yau: $\mathbb{CP}^{3|4}$ has weightless super volume form

$$D^{3|4}Z = D^3 Z \,\mathrm{d}\chi_1 \dots \mathrm{d}\chi_4 \in \Omega_{Ber}.$$

'Super-Ward' for $\mathcal{N} = 4$ SYM:
A dbar-op $\bar\partial_A = \bar\partial_0 + A$ on bundle over $\mathbb{CP}^{3|4}$ has expansion

$$A = a + \chi_a\psi^a + \chi_a\chi_b\phi^{ab} + \chi^{3a}\tilde\psi_a + \chi^4 b$$

and $\bar\partial_A^2 = 0 \;\leftrightarrow$ solns to self-dual $\mathcal{N} = 4$ SYM on space-time.

Action for fields with self-dual interactions:[Sokatchev, Witten]
SD interactions $\leftrightarrow$ holomorphic Chern-Simons action

$$S_{sd} = \int_{\mathbb{PT}} \mathrm{tr}(A \wedge \bar\partial A + \frac{2}{3}A^3)_\wedge D^{3|4}Z\,.$$

Incorporating interactions of full $\mathcal{N} = 4$ SYM

Nair, M., Boels, Skinner, 2006

Extension to full SYM:

$$S_{full}[A] = S_{sd}[A] + S_{int}[A]$$

includes non-local interaction term:

$$\begin{aligned} S_{int}[A] &= g^2 \int_{\mathbb{M}} \mathrm{d}^{4|8}\mathbf{x} \; \log \; \det(\bar{\partial}_A|_X) \\ &= g^2 \sum_{n=2}^{\infty} \frac{1}{n} \int_{\mathbb{M}\times X^n} \mathrm{d}^{4|8}\mathbf{x} \frac{\mathrm{tr}\,(A_1 A_2 \ldots A_n)\, D\sigma_1 \ldots D\sigma_n}{\langle\sigma_1\sigma_2\rangle\langle\sigma_2\sigma_3\rangle \ldots \langle\sigma_n\sigma_1\rangle} \end{aligned}$$

$X = \mathbb{CP}^1_{\mathbf{x}} \subset \mathbb{PT}$ for $\mathbf{x} \in \mathbb{M}^{4|8}$, $\sigma_i \in X_i$, i^{th} factor,

$$A_i = A(Z(\sigma_i))\,, \qquad \text{and} \qquad K_{ij} = \frac{D\sigma_j}{\langle\sigma_i\,\sigma_j\rangle}$$

is Cauchy kernel of $\bar{\partial}^{-1}$ on X at σ_i, σ_j.

Axial gauge Feynman rules

Choose 'reference twistor' Z_*, impose gauge: $\bar{Z}_* \cdot \frac{\partial}{\partial \bar{Z}} \lrcorner A = 0$.

Payoff: Cubic Chern-Simons vertex $= 0 \rightsquigarrow$ Feynman rules:

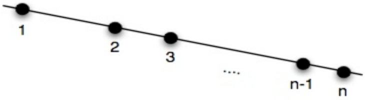
- Propagator = delta-function forcing Z, Z', Z_* to be collinear.

$$\Delta(Z, Z') = \frac{1}{2\pi i}\bar{\delta}^{2|4}(Z, Z_*, Z') := \frac{1}{2\pi i}\int \frac{\mathrm{d}c\,\mathrm{d}c'}{cc'}\bar{\delta}^{4|4}(Z_* + cZ + c'Z')$$

- log-det term gives 'MHV vertices':

$$V(Z_1, \ldots, Z_n) = \int_{\mathbb{M}\times X^n} \frac{\mathrm{d}^{4|4}Z_A \mathrm{d}^{4|4}Z_B}{Vol\ GL(2)} \prod_{r=1}^{n} \frac{\bar{\delta}^{3|4}(Z_r, Z_A + \sigma_r Z_B)}{(\sigma_{r-1} - \sigma_r)} d\sigma_r \,.$$

Vertices force $Z_1, \ldots Z_n$ to lie on line $X = \{Z(\sigma) = Z_A + \sigma Z_B\}$



Simplicity: # propagators $=$ MHV degree $+2\times$# loops.

Amplitudes: Transform rules to momentum space $\rightsquigarrow$ 'MHV rules' [CSW 2005] where vertices $=$ off-shell MHV amplitudes.

But MHV diagrams also give Wilson loops etc.:

Null polygons: Supermomentum conservation for colour ordered momenta $\rightsquigarrow$ null polygon $\{\mathbf{x}_i\} = \{(x_i, \theta_i)\} \subset \mathbb{M}$:

$$(p_i^{AA'}, \eta_i^a \lambda^A) = (x_i^{AA'} - x_{i+1}^{AA'}, \theta_i^A - \theta_{i+1}^A)\,.$$

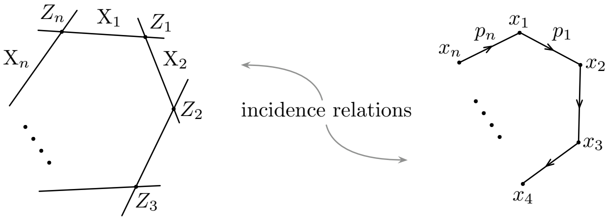
Conjecture (Alday, Maldacena)

Let $W(x_1, \ldots, x_n) =$ Wilson-loop around momentum polygon.

$$\textit{All loop MHV amplitude} = \textit{MHV tree} \times \langle W(x_1, \ldots, x_n)\rangle.$$

Momentum polygons in twistor space

Generic polygon in $\mathbb{PT}$ [Hodges] $\leftrightarrow$ null polygon in space-time.



Change variables so that $(X_1, \ldots, X_n) = (Z_1, \ldots, Z_n)$.

Important simplification: $Z_i \in \mathbb{PT}$ are unconstrained.

What is Wilson loop in $\mathbb{PT}$?

For Wilson-loop, need holonomy around polygon in $\mathbb{PT}$.

- Vertices Z_i,
- Edges $X_i = \{Z(\sigma) = \sigma Z_{i-1} + Z_i, \sigma \in \mathbb{C} \cup \infty\}$.
- Global frame $F_i(\sigma)$ of $E|_{X_i}$ on X_i with

$$\bar\partial_{\mathcal{A}}|_{X_i} F_i(\sigma) = 0, \qquad F_i(\infty) = F_i|_{Z_{i-1}} = 1\,.$$

- Perturbatively iterate $F_i = 1 + \bar\partial^{-1}(\mathcal{A}F_i)$ to get

$$F_i = 1 + \sum_{r=1}^{\infty} \prod_{s=1}^{r} \bar\partial^{-1}_{s-1\,s}\mathcal{A}(\sigma_s)\,, \qquad (\bar\partial^{-1}_{r\,s} f)(\sigma_r) = \int_{L_{X_i}} \frac{f(\sigma_s)\,\mathrm{d}\sigma_s}{\sigma_r - \sigma_s}$$

- Define

$$W = \operatorname{tr} \prod_{i=1}^{n} F_i|_{Z_i} = \operatorname{tr} \prod_{i=1}^{n} F_i(0)\,.$$

Agrees with space-time Wilson loop on-shell.

The S-matrix as a holomorphic Wilson loop

Theorem (Bullimore, M., Skinner, 2010-11)

For planar $\mathcal{N} = 4$ SYM:

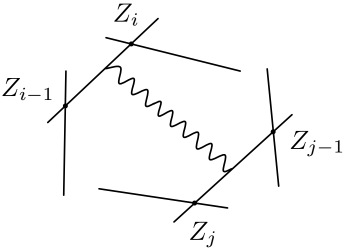
Amplitude loop-integrands = (holomorphic) Wilson loop integrand

$$\mathcal{A}(1, \dots, n) = \langle W(Z_1, \dots, Z_n) \rangle \mathcal{A}^{tree}_{MHV} \, .$$

- *Tree amplitudes $\leftrightarrow$ Wilson-loop in self-dual sector (g=0).*
- *Loop expansion for $\mathcal{A}$ = g-expansion for W.*
- *The Axial gauge twistor space diagrams for amplitude are planar duals of those for Wilson-loop correlator.*

Proof: comparison of Feynman rules on $\mathbb{PT}$ (or BCFW).

NMHV case: for A^2 part of $\langle W\rangle$, $\langle A(Z)A(Z')\rangle = \Delta(Z,Z')$: gives $\langle W\rangle = \sum_{i<j}\Delta_{ij}$ where $\Delta_{ij} =$



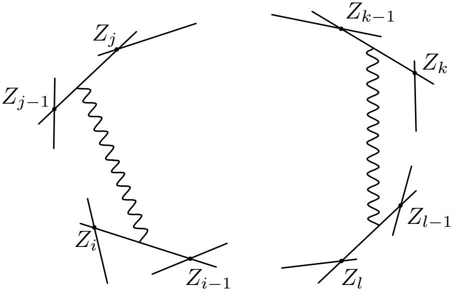
$$= [\,*\,, i-1, i, j-1, j]\,,$$

$$[1,2,3,4,5] \quad := \quad \int \frac{\mathrm{d}c_1\mathrm{d}c_2\mathrm{d}c_3\mathrm{d}c_4\mathrm{d}c_5}{c_1c_2c_3c_4c_5}\frac{1}{\mathrm{Vol\,Gl}(1)}\;\bar{\delta}^{4|4}\left(\sum_{i=1}^{5} c_i Z_i\right)\,,$$

$$= \quad \frac{\prod_{a=1}^4\left((1234)\chi_5^a + \text{cyclic}\right)}{(1234)(2345)(3451)(4512)(5123)}$$

is the 'R-invariant'.

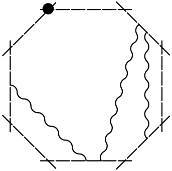
N^{2}MHV: quartic terms in A in W give two Wick contractions



No crossed propagators for planarity.

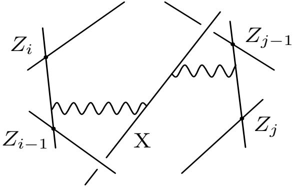
N^{k}MHV tree amplitudes $\leftrightarrow$ k propagators $\rightsquigarrow$ k R-invariants.

But now have 'Boundary diagrams' e.g.



Loops

At MHV with one MHV vertex obtain $\sum_{i,j} K_{ij}$ with $K_{ij} =$



$$= \int_\Gamma \mathrm{D}^{3|4} Z_A \wedge \mathrm{D}^{3|4} Z_B \, [\,*\,, i-1, i, A, B'][\,*\,, j-1, j, A, B'']$$

Loop momenta $\leftrightarrow$ location of line $X = \langle Z_A \, Z_B \rangle$. Recall:

$$[*, i-1, i, A, B] := \int \frac{\mathrm{d}c_1 \mathrm{d}c_2 \mathrm{d}c_3 \mathrm{d}c_4}{c_1 \, c_2 \, c_3 \, c_4} \, \bar{\delta}^{4|4}(Z_* + c_1 Z_A + c_2 Z_B + c_3 Z_{i-1} + c_4 Z_i)$$

can integrate $\frac{D^{4|4} Z_A \wedge D^{4|4} Z_B}{vol \, GL_2}$ against delta functions

$$K_{ij} = \frac{1}{(2\pi i)^2} \int \frac{\mathrm{d}c_0 \mathrm{d}c_1 \mathrm{d}b_0 \mathrm{d}b_1}{c_0 c_1 \, b_0 b_1} \, .$$

External data encoded in integration contour (see later).

Loop integrands and correlators

Lagrangian insertions: $S_{int}[\mathcal{A}] = \int d^4x \mathcal{L}_{int}(x)$ where

$$\mathcal{L}_{int}(x) = \int d^8\theta \; \log\det(\bar{\partial}_{\mathcal{A}}|_X)\,.$$

Forming tree correlator with ℓ insertions gives loop integrand

$$\langle W(\mathbf{x}_1,\ldots,\mathbf{x}_n)\mathcal{L}(x_1)\ldots\mathcal{L}(x_l)\rangle_{tree}\,.$$

Super BPS operators: it is invariant to integrate over just 4 θs

$$\mathcal{O}(x,\theta,Y) := \mathrm{tr}(\Phi(\mathbf{x})\cdot Y)^2 = \int (Y^{ab} d\theta_{a\alpha} d\theta^{\alpha}_{b})^2 \log\det(\bar{\partial}_{\mathcal{A}}|_X)$$

where $Y^{[ab}Y^{cd]} = 0$ (so depends on the 4 θs with $\theta_{a\alpha}Y^{ab} = 0$).

Proposition (Alday, Eden, Korchemsky, Maldacena, Sokachev, Heslop, Adamo, Bullimore, M., Skinner)

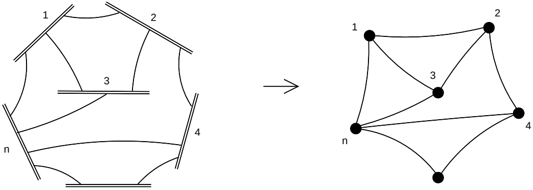
$$\lim_{(x_i-x_{i+1})^2\to 0} \left\langle \prod_{i=1}^{n} \frac{(x_i-x_{i+1})^2}{Y_i\cdot Y_{i+1}} \mathcal{O}(\mathbf{x}_i, Y_i) \right\rangle = \langle W(\mathbf{x}_1,\ldots,\mathbf{x}_n)\rangle^2\,.$$

Gives 'triality' Amplitude $\leftrightarrow$ Wilson loop $\leftrightarrow$ BPS correlator.

MHV diagrams for correlators

[Chicherin, Doobary, Eden, Heslop, Korchemsky, M., Sokatchev]

- Obtain diagrams in twistor space for $\langle \mathcal{O}(\mathbf{x}_1) \ldots \mathcal{O}(\mathbf{x}_n) \rangle$.



- Double line $\leftrightarrow$ line $X_l = \langle Z_{l1}, Z_{l2} \rangle \subset \mathbb{PT} \leftrightarrow \mathbf{x}_i$.
- Solid lines $\leftrightarrow$ twistor propagators.
- As $(x_i - x_{i+1})^2 \to 0$, consecutive lines join.
- Diagram $\to 0$ unless consecutive lines are connected by propagator $\rightsquigarrow$ Wilson loop.

The momentum twistor Grassmannian

Recall from Nima's talks:

Grassmannian contour integral formula:[M. & Skinner (2009)]
For $C_{ri} \in G(k,n)$ consider

$$\oint_{\Gamma_{4k}} \frac{d^{nk}C \prod_{r=1}^{k} \bar{\delta}^{4|4}(C_{ri}Z_i)}{vol\,GL(k)(1\ldots k)(2\ldots k+1)\ldots(n\,1\ldots k-1)}\,.$$

Contour-integrate down to $4k$-cycles and fix remaining parameters against delta-functions
$\rightsquigarrow$ terms in BCFW expansion (and leading singularities).

Loop integrands: include L lines $(Z_{A_1}, Z_{B_1}), \ldots (Z_{A_L}, Z_{B_L})$ in $(Z_1, \ldots, Z_n) \rightsquigarrow$ extend to $G(k+2L, n+2L)$.

$4k+8L$-cycles are characterized as positive cells.

Wilson loop diagrams in positive Grassmannian

- With $k+2L$ propagators diagram gives

$$\prod_{r=1}^{k+2L} \int_{\mathbb{CP}^4} \frac{\mathrm{d}c_{r\,0}\mathrm{d}c_{r\,i_1^r}\mathrm{d}c_{r\,i_2^r}\mathrm{d}c_{r\,i_3^r}\mathrm{d}c_{r\,i_4^r}}{Vol\;GL(1)\;c_{r\,0}c_{r\,i_1^r}c_{r\,i_2^r}c_{r\,i_3^r}c_{r\,i_4^r}} \bar{\delta}^{4|4}\left(Y_r\right)\,,$$

  where $\qquad Y_r = c_{r\,0}Z_* + \sum_{p=1}^{4} c_{r\,i_p^r} Z_{i_p^r}$

- Taking this into account we can write

$$Y_r = \sum_{i=1}^{n} c_{r0}Z_* + C_{ri}Z_i \qquad C_{ri} = C_{ri}\left(c_{r\,i_p^r}\right)$$

  The $C_{ri}(c_{ri_p^r})$ define $3k$-cycle in $Gr(k+2L, n+2L)$.
- c_{r0} provides k extra parameters to get to $4k$-cycle.
- $4k$-cycles have d-log volume forms in parameters $c_{ri_p^r}$.

Proposition (Agarwala,Marin-Amat)

The $3k$-cycles are $3k$-cells in the positive Grassmannian.

Positive Grassmannians questions

- Is d-log parametrization for $4k$-cycle determined by positive $3k$-cell?
- What is correspondence between Wilson-loop diagram and standard classification of positive cells?
- **Key question:** Do all $3k$-cells arise? If not, how can we characterize those that do?

See Susama's talk for more on these questions.

Conjecture

The $3k$-cells that arise are those that project to $3k$-cells in the amplituhedron.

Bosonization, positivity and 'hedronizing

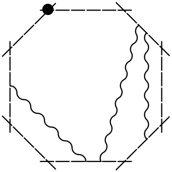
- Take real and bosonize fermionic parts of Y, Z by

$$\mathbb{C}^{4|4} \to \mathbb{R}^{4+k} \quad \text{with} \quad Z^r = \chi \cdot \phi^r, \quad r = 1, \ldots, k\,.$$

- The rth fermionic delta-function arises by

$$\delta^{0|4}(\chi) = \int (Y^r)^4 \, \mathrm{d}^4 \phi^r\,.$$

- Take data $\{Z_1, \ldots, Z_n\}$ positive.
- Planarity $\Rightarrow$ positivity of $c_{r\, i_p^r} \Rightarrow C_{ri} \in Gr_+(k, n)$ encoding planarity of 'Boundary diagrams' such as:



Tiling Amplituhedra/Wilsonohedra/Correlahedra

w/ Agarwala, Eden, Heslop, following Arkani-Hamed, Hodges, Trnka

- Positive $c_{r\,i^r_p}$ gives $4k$-dimensional tiles in $Gr(k, k+4)$

$$Y_r = c_{r0} Z_* + C_{ri} Z_i \,.$$

where $C = C\left(c_{r\,i^r_p}\right)$ and $c_{r0} = 0 \rightsquigarrow$ positive $3k$-cell.

- Can we tile amplituhedra? Correlahedra?

$$\begin{aligned} 0 &< \langle Y_1 \dots Y_{k+2L} Z_{i-1} Z_i Z_{j-1} Z_j \rangle \\ 0 &< \langle Y_1 \dots Y_{k+2L} Z_{l1} Z_{l2} Z_{j-1} Z_j \rangle \\ 0 &< \langle Y_1 \dots Y_{k+2L} Z_{l1} Z_{l2} Z_{m1} Z_{m2} \rangle \end{aligned}$$

- Above gives $\langle W \rangle^2$ (cf correlator $\leftrightarrow$ Wilson-loop).
- Unlike BCFW, tiles lie both inside and out for $k \geq 2$.
- Spurious boundaries cancel (subtle for bdy diagrams).
- Gives 'hedra formulation for correlators.

[Work in progress.]

Summary & conclusions

- Geometry of amplituhedra and Grassmannians is built into Feynman rules of twistor action in axial gauge.
- Framework extends to more general correlahedra.
- MHV diagram tiling is imperfect with tiles crossing in and out of correlahedra.
- Need to turn it into a better oiled machine for the actual integrals (unitarity, motives, symbols, cluster algebras, Fuchsian differential equations, integrability, regularization....).

Thank You!