# Multi-Layered Baby Skyrmions

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## Introduction

The baby Skyrme model is a 2-dimensional analogue of the Skyrme model, which is a (3+1)-dimensional model for pions, first introduced by T.H. Skyrme (1982), which is a low energy effective model for the high quark color limit of quantum chromodynamics (QCD). Furthermore, the model can be considered as a stabilised version of the $O(3)$ sigma model, whereby the introduction of a Skyrme term and potential term to balance to stop the soliton from expanding infinitely to fill the entire space, or shrinking to a localised point.

This poster studies the baby Skyrme model under the potential with 2 vacua: $V(\phi) = (1-\phi_3^2)$, which was first introduced in Weidig(1999). We explore a new family of solutions of the baby skyrme model which admits a dihedral symmetry, which has not previously been studied in this model.

## Model

We seek only static solutions of the equations of motion, which are obtained from varying the equations of motion with respect to some field $\phi$. We obtain the equations of motion by calculating the variation of some energy functional, since we seek to minimize this energy functional for the field $\phi$.

$\phi$ is a map between such that: $\phi : S^2 \mapsto \mathbb{R}^2$. We represent $\phi$ by a 3 component unit vector, which is normalised such that: $\phi \cdot \phi = \phi_1^2 + \phi_2^2 + \phi_3^2 = 1$, such that the target space is constrained from $R^2$ to $S^2$. Furthermore, a one point compactification of the domain space defines a point (the vacuum) at $\pm\infty$. By including a point at infinity, the domain becomes $\mathbb{R}^2 \cup \{\infty\}$, which is equivalent to the unit 2−sphere, hence $\phi$ is a map between two 2−spheres: $\phi : S^2 \mapsto S^2$.

The static energy functional that we study is given:

$$E = \frac{1}{4\pi}\int [\frac{1}{2}(\partial_x\phi^2 \ + \ \partial_y\phi^2) \ + \ \frac{1}{2}(\partial_x\phi \ \times \ \partial_y\phi)^2 + \mu^2(1-\phi_3^2)]d^2x, \tag{1}$$

where $\phi$ is as described above, and $(x, y)$ are the spatial coordiantes for the domain. We choose $\mu^2 = 0.1$ to be consistent with the literature. This choice of potential depends only on $\phi_3$, hence we break the $O(3)$ symmetry of the model to an $O(2)$ symmetry. However, the family of solutions we explore also admit a $D_N$ dihedral symmetry, as seen in the results section.
An example of previously studied solutions is in Weidig(1999), which admit an axial symmetry for all charges. These solutions are given by axially symmetric ansatz:

$$\phi = (\phi_1, \phi_2, \phi_3) = (\sin(f)\cos(B\theta), \sin(f)\sin(B\theta), \cos(f)), \tag{2}$$

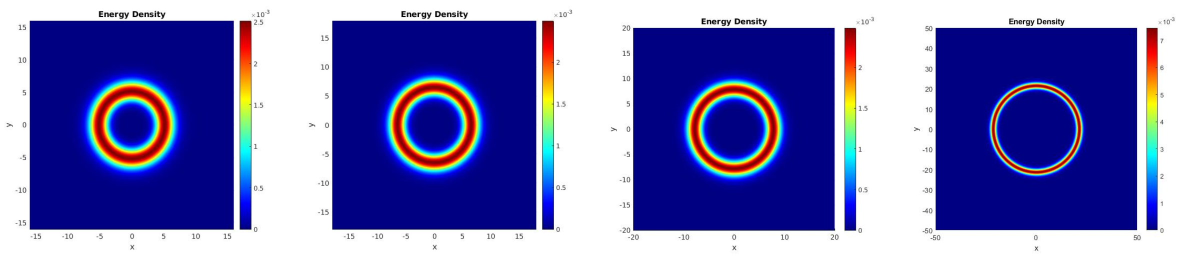


**Figure 1:** Energy density contour plots for the standard axially symmetric solutions for charges $B = 4$, $B = 5$, $B = 6$ and $B = 16$ respectively, plotted against the domain.

The motivation behind the following initial condition is to study solutions with nested rings, which is later extended to solutions with multiple layers. We do this in order to find a more interesting solution rather than an axial ring for all charges.
We consider an initial condition that allows for these layers of rings to exist, whereby $\phi_1$, $\phi_2$ and $\phi_3$ are set to the following functions:

$$\phi = \begin{cases} (\sin f\cos(-N_1\theta), \ \sin f\sin(-N_1\theta), \ \cos f), & r < r_0, \\ (\sin f\cos(N_2\theta), \ \sin f\sin(N_2\theta), \ \cos f), & r_0 \le r \le r_1, \\ (\sin f\cos\theta, \ \ \sin f\sin\theta, \ \ \cos f), & r \ge r_1. \end{cases}$$

where the shape function winds around the target space for the number of layers. Namely, we create a inner ring of topological charge $N_1$, then an outer ring of topological charge $N_2$ which winds in the opposite direction.

Furthermore, $(r, \theta)$ are polar coordinates, $r_0$, $r_1$ are constants chosen such that there is enough space for the ring to exist in the space, and $f(r)$ is a profile function such that:

$$f(r) = \begin{cases} 2\pi - \frac{\pi r}{r_0}, & r < r_0, \\ \pi - \frac{\pi(r-r_0)}{r_1 - r_0}, & r_0 \le r \le r_1, \\ 0, & r \ge r_1. \end{cases}$$

where the variables and constants are the same as that for the definition of $\phi$.

It is worth noting that the topological charge separates the solution spaces into 2 disconnected manifolds, however the solutions are obtained via a gradient flow algorithm with a $4^{th}$ order finite difference scheme, on a grid of size $601 \times 601$, with lattice spacing 0.1., hence the manifolds are connected.

## Results

At higher charges, as the previously understood solutions, which are the axial rings grow larger, the cost of the curvature of the rings decreases, resulting in an exponential tail in the energy per baryon values, which can be seen in figure 5.
Below are some double ring solution. We differentiate between the solutions of the same overall topological charge denoting the inner and outer charge, such that a charge $N$ solution will be stated as a $(N_1, N_2)$ solution, where $N_1$ is the topological charge of the inner ring, and $N_2$ is the topological charge of the outer ring.

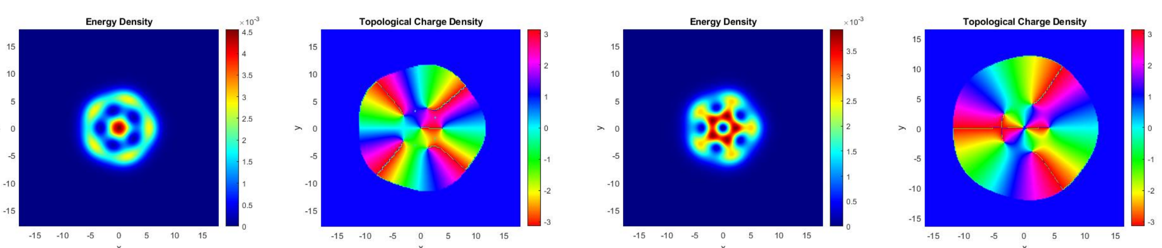


**Figure 2:** Energy density and phase coloured plots respectively for solutions with topological charge 5, $(1, 4)$, and $(2, 3)$ respectively. The phase is $\psi = \tan^{-1}(\frac{\phi_2}{\phi_1})$.

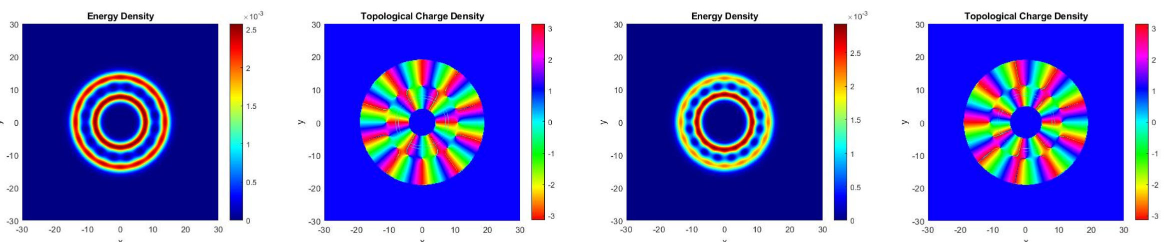


**Figure 3:** The left hand side shows energy density for (6,10) and (7,9) configurations respectively. The right hand side shows the topological charge density coloured by the phase, $\psi = \tan^{-1}(\frac{\phi_2}{\phi_1})$.

In this example, the $(1, 4)$ solution has a lower energy than the $(2, 3)$ solution. A comparison of the solutions at different topological charges can be seen in figure 5.

For the double ring solutions, the size of the solution is smaller, hence the cost of the curvature for an axial solution of the same size is higher, however, we have an attraction between the layers pulling them together. This results in a dihedral symmetry, since the attraction depends on whether the inner and outer rings are in or out of phase with each other. Moreover, the two rings move in and out of phase with each other at various points, creating lumps of energy.

At low topological charges, these solutions are a local minima, however, at higher topology charges, these solutions could provide a global minima solutions. We find that a local minima of these new configurations comes from a balance between the layers of the solution, such that the charge of the inner ring is less than or equal to half that of the outer ring.

The cost of the curvature for these double ring solutions is clearly higher than that of the previously understood solutions, however, at higher charges with rings separated appropriately, this difference cost could be negligible, and the attraction between the rings could lower the energy such that the solutions may be a global minima.

## Comparison

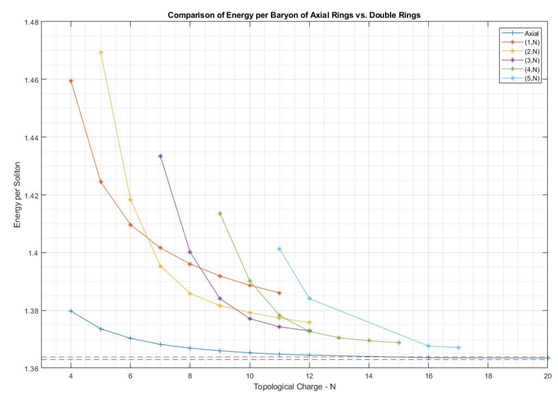


**Figure 4:** Energy per baryon of axial configuration vs. different double ring configurations.

## Rings with a Higher Count of Nested L

The following solution is has 4 nested layers. The total topological charge is $B = 24$, however the layers have topological charge: $(2, 4, 8, 10)$. The solution exhibits a $D_6$ symmetry, which appears to be a result of the total charge between neighbouring rings. In this example, the inner two rings have total topological charge $B = 6$, and the outer two rings have total topological charge $B = 18$.
As seen with the double rings, these solutions should have $D_6$ and $D_18$ symmetry respectively, however, as they are nested, we appear to get a $D_6$ symmetry, which is the greatest subgroup between the two nested double rings.

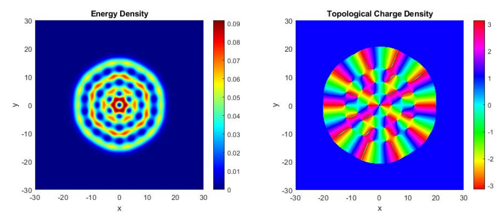


**Figure 5:** Energy density plot of a (2,4,8,10)-quad ring, with energy per baryon $E = 1.3901$.

## Symmetries

We plot the phase with respect to the orientation of a charge $(2, 3)$ ring. This suggests that there are 5 localised points where the energy is minimal which suggests the dihedral $D_5$ symmetry.

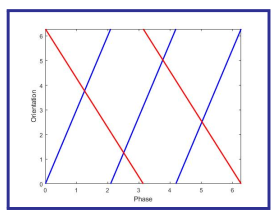

**Figure 6:** The phase of the inner ring in blue, and outer ring in red.

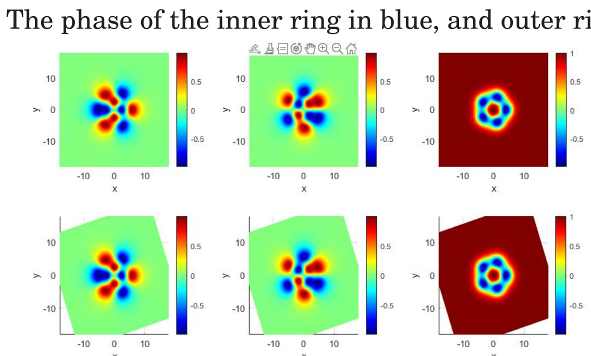

**Figure 7:** $(\phi_1, \phi_2, \phi_3)$ original, then rotated in domain and then back in target space for a $(2, 3)$ configuration.

We verify the symmetry of the solutions by looking at the fields. Here we show that a rotation on the target space and a rotation in the domain in the opposite direction leaves the solution invariant.

For double rings we confirm this rotation to be a $SO(3)$ rotation matrix of $2\pi/N$, where $N$ is the total topological charge of the solution, fixed around the $\phi_3$ axis, performed $N_1$ times.
Note, we can also confirm the reflection symmetry (not shown here).

## Conclusion

We have shown a new family of solutions for this model which are local minima, and admit different symmetries to those previously presented.

This work could be extended using by considering an ansatz using rational maps which exhibit the same symmetries that have been observed in this poster. This will allow us to seek certain solutions at higher charges, without having a high number of layers.

At higher charges the solutions collapse to have more layers. It is hopeful that at higher charges a global minima could present itself with a higher number of rings, once configured appropriately.

## References

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