Applications of Differential Geometry to Mathematical Physics

This course is hosted and taught by members of staff of the University of Kent and will take place at the campus in Canterbury. Andy Hone and Steffen Krusch are focussing more on the physical applications, Gavin Brown is teaching the pure mathematics parts. Jim Shank and other members of staff will assist during the example classes.

1 Philosophy of the course

The main aim of the course is to introduce our PhD students to important concepts in differential geometry and provide a dictionary between physics and mathematics (gauge theory and fibre bundles). During the course, practical examples will have priority over proofs. Connections and overlaps between the different lectures are very much encouraged.

Since the course covers a large amount of material, printed lecture notes will be made available and there will be four example classes during which members of staff will be available to assist. The example classes are intended to reinforce the material and provide the opportunity to go into more depth. Therefore, there will be a rather large selection of exercises and students are encouraged to choose the problems according to their prior knowledge and their interests.

2 Preliminary Schedule

There will be six lectures and four example classes.

2.1 Lecture 1

This lecture gives an overview of various concepts in differential geometry and will be very much example-led. The assumption is that many students will have been exposed to at least some of the material.

We start by discussing manifolds and illustrate this concept by constructing charts for S^2 and $SU(2) \cong S^3$. Then we introduce the concept of fibre bundles starting with the trivial bundle $E = B \times F$. We discuss principal G bundles using $G = U(1)$ and $G = SU(2)$ as illustrating examples. This will make it easier to show connections to Yang-Mills theory. We will discuss the Hopf bundle in detail using the charts we have derived earlier to write down the relevant transition functions.

We also introduce the idea of a vector field and define the tangent bundle (example: tangent bundle of S^2). Finally, we give an outline of the ingredients of General relativity, namely, define a connection, metric, Levi-Civita connection, the torsion and the curvature tensor and present Einstein's equations.

2.2 Lecture 2

The second lecture discusses some of the mathematical concepts in more detail. Differential forms are introduced and then used to define connections in a more formal setting. Further topics might include line bundles, de Rham cohomology, characteristic classes or homotopy theory.

2.3 Lecture 3

This lecture gives an introduction to important concepts in physics, starting with Lagrangians in classical mechanics, and the tangent bundle to configuration space. Examples will include the dynamics of particles in curved space-time. Next we describe Legendre transformations and the cotangent bundle to configuration space, Hamiltonian mechanics, symplectic geometry, canonical coordinates (Darboux theorem) and Poisson brackets. Liouville's theorem on finite-dimensional integrable systems will also be presented, with examples.

2.4 Lecture 4

Here we discuss Lagrangians in field theory: scalar fields, electromagnetism and Yang-Mills theory leading to the Standard Model of particle physics. Yang-Mills-Higgs theory with U(1) and SU(2) gauge fields will be covered in more detail, with a dictionary between gauge theory and the theory of fibre bundles. Zero curvature connections and integrable systems in infinite dimensions will be treated, using sine-Gordon field theory as the main example.

2.5 Lecture 5

This lecture will introduce complex manifolds, Kähler manifolds and discuss the interplay of, for example, symplectic structure, complex structure and the metric. Further topics are K3-surfaces and Calabi-Yau 3-folds, both from a local and global perspective.

2.6 Lecture 6

This lecture will discuss the application of homotopy theory to the "classification" of topological solitons and will discuss kinks, vortices and maybe monopoles. We will also explain the moduli space approximation, namely how the low energy dynamics of topological solitons can be approximated by geodesic motion on the (moduli) space of static minimal energy solutions. The geometry of this moduli space plays an important role for our understanding of the dynamics of solitons. In fact, the N-vortex moduli space has a Kähler structure.

3 Recommended Reading

The lectures are inspired by selected chapters of the following books.

- M Nakahara, *"Geometry, Topology and Physics"*, Second Edition, Graduate Student Series in Physics, Institute of Physics Publishing, 2003
- R Bott and LW Tu, *"Differential Forms in Algebraic Topology"*, Springer Verlag, New York, 1982
- NS Manton and PM Sutcliffe, *"Topological Solitons"*, Cambridge University Press, 2004
- VI Arnold, *"Mathematical Methods of Classical Mechanics"*, Second Edition, Graduate Texts in Mathematics, Springer Verlag, New York, 1997
- O Babelon, D Bernard and M Talon, *"Introduction to Classical Integrable Systems"*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 2003
- D Huybrechts, *"Complex geometry"*, Springer-Verlag, Berlin, 2005