LTCC Geometry and Physics: Exercise Sheet 3

1. Let M, J be an almost complex manifold and g be a hermitian metric. Let X be a vector of length 1: $g(X, X) = 1$. Show that X, JX is an orthonormal pair—that is, $g(JX, JX) = 1$ and $g(X, JX) = 0$.

2. If M is a complex manifold, define an almost complex structure on a patch with coordinates $z^j = x^j + iy^j$ by $J(\frac{\partial}{\partial x^j} = \frac{\partial}{\partial y^j})$ and $J(\frac{\partial}{\partial y^j} = -\frac{\partial}{\partial x^j})$. Show that this rule is unchanged by holomorphic change of coordinates so that it defines an almost complex structure on the whole of M. (Use the Cauchy–Riemann equations, as usual.)

3. Let X and Y be the stereographic coordinates via projection from the south pole. Invert this projection to show that points on $S^2 \subset \mathbb{R}^3$ are given by

$$\mathbf{n} = \frac{1}{1 + X^2 + Y^2} \begin{pmatrix} 2X \\ 2Y \\ 1 - X^2 - Y^2 \end{pmatrix}.$$

   Let

$$\mathbf{n}_X = \frac{\partial \mathbf{n}}{\partial X} \quad \text{and} \quad \mathbf{n}_Y = \frac{\partial \mathbf{n}}{\partial Y}.$$

   Show that $\mathbf{n}_X$ and $\mathbf{n}_Y$ are orthogonal to each other and to $\mathbf{n}$ and hence span the tangent space T_pM for fixed p. Show that $(\mathbf{n} \times .)$ gives an almost complex structure on T_pM.

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