LTCC Geometry and Physics: Exercise Sheet 2

(Note: Unless otherwise stated, the Einstein summation convention is assumed throughout.)

1. Given a Riemannian manifold M with metric g, consider the action $S = \int_{t_0}^{t_1} L\,dt$ with the purely kinetic Lagrangian

$$L = \frac{1}{2} g(\dot{\mathbf{x}}, \dot{\mathbf{x}}) = \frac{1}{2} g_{jk} \dot{x}^j \dot{x}^k, \tag{1}$$

where $g = (\, g_{jk}(\mathbf{x})\,)$ is the metric given in terms of local coordinates x^j for a point $\mathbf{x} \in M$ with tangent vector $\dot{\mathbf{x}}$; the dot denotes differentiation w.r.t. the parameter t along a curve.
(i) Applying the principle of least action, $\delta S = 0$, calculate the Euler-Lagrange equations for this Lagrangian, and show that they can be written in the form

$$\ddot{x}^j + \Gamma^j_{kl} \dot{x}^k \dot{x}^l = 0, \tag{2}$$

where Γ^j_{kl} (the Christoffel symbols) should be found in terms of g and its derivatives.
(ii) Perform a Legendre transformation and hence reformulate the geodesic equations as an Hamiltonian system on T^*M.

2. Write down the Euclidean metric on $\mathbb{R}^3$ in Cartesian coordinates, and the invariant volume form. Use this to calculate the invariant 2-form on the sphere S^2, and show that this is a symplectic form.

3.(i) For a pair of vector fields X, Y on a smooth manifold M of dimension d, define their commutator $[X, Y]$ (or Lie bracket) by the commutator of the corresponding differential operators. Show that with this definition the vector fields form a Lie algebra.
(ii) Now suppose that $d = 2n$ and (M, ω) is a symplectic manifold. For Hamiltonian vector fields X_G, X_H corresponding to smooth functions G, H, show that the following formula holds:

$$[X_G, X_H] = -X_{\{G,H\}},$$

where $\{,\}$ denotes the Poisson bracket. (Hint: Use Darboux's theorem.)
(iii) Prove that the Hamiltonian vector fields form a Lie subalgebra of the vector fields.

4.(i) Check that the formula

$$\{\pi_j, \pi_k\} = -\epsilon_{jkl} \pi_l$$

defines a Poisson bracket on $\mathbb{R}^3$ with coordinates (π_1, π_2, π_3), and show that (on linear functions) this is isomorphic to the Lie algebra $\mathfrak{so}(3)$ of the rotation group in three dimensions.
(ii) Write down the Poisson tensor for the above bracket and calculate its rank. Show that $C = \pi_1^2 + \pi_2^2 + \pi_3^2$ is a Casimir for this bracket, and describe the symplectic leaves.
(iii) Consider the Hamiltonian system on $\mathbb{R}^3$ defined by the Hamiltonian

$$H = \frac{\pi_1^2}{2I_1} + \frac{\pi_2^2}{2I_2} + \frac{\pi_3^2}{2I_3}.$$

Write down the equations of motion, and show that they are equivalent to the Euler top (free motion of a rigid body about a fixed point) in terms of the angular momentum $\boldsymbol{\pi}$ and angular velocity $\boldsymbol{\omega}$, where $\boldsymbol{\pi} = I\boldsymbol{\omega}$ with the inertia tensor $I = \mathrm{diag}(I_1, I_2, I_3)$. Can you describe the orbits?

Professor Andy Hone, October 2010.