Differential Geometry and Soliton Dynamics

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Outline

- Homotopy theory
- Ginzburg-Landau vortices
- Vortices at critical coupling and the vortex moduli space
- Relativistic vortex dynamics
- First order vortex dynamics
- Vortices in different geometries

The Fundamental group $\pi_1(M)$

- Given a manifold M and an interval $I = [0, 1]$ we can define *paths*

$$\alpha : I \to M : t \mapsto \alpha(t), \text{ where } \alpha(0) = p_0, \ \alpha(1) = p_1.$$

- A *loop* is a path with $p_0 = p_1$.
- Paths can be multiplied via

$$\alpha * \beta(s) = \begin{cases} \alpha(2s) & 0 \leq s \leq \frac{1}{2}, \\ \beta(2s-1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

- The constant path is $c(s) = p_0$ for all $s \in I$.
- The inverse of a paths is $\alpha^{-1}(s) = \alpha(1-s)$.
- This is not a group, yet!

The Fundamental group II

Homotopy

Let $\alpha, \beta : I \to M$ be loops at p_0.
α and β are *homotopic*, $\alpha \sim \beta$, it there exists a continuous map $F : I \times I \to M$ such that

- $F(s,0) = \alpha(s)$ and $F(s,1) = \beta(s)$ for all $s \in I$.
- $F(0,t) = F(1,t) = p_0$ for all $t \in I$.

- $\alpha \sim \beta$ is an equivalence relation.
- Let $[\alpha]$ be the equivalence class given by α.
- Define a product on equivalence classes by $[\alpha] * [\beta] = [\alpha * \beta]$.
- This gives the *fundamental group* $\pi_1(M, p_0)$.[1]
- Examples: $\pi_1(S^1) = \mathbb{Z}, \quad \pi_1(\mathbb{R}^2 \setminus \{0\}) = \mathbb{Z}, \quad \pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$.
- Note $\pi_1(M \times N) = \pi_1(M) \oplus \pi_1(N)$.

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[1] If M is arcwise connected then $\pi_1(M, p_0)$ is isomorphic to $\pi_1(M, p_1)$.

Higher Homotopy groups $\pi_n(M)$

- This generalizes naturally to higher homotopy groups: Consider maps from the cube $I^n = I \times \cdots \times I$ to a manifold M such that all the points on the boundary ∂I^n of the cube are mapped to $p_0 \in M$:

$$\alpha : (I^n, \partial I^n) \to (M, p_0).$$

- Again we can form the product $\alpha * \beta$ and define the equivalence classes $[\alpha]$ (also known as homotopy classes).
- This gives us the nth homotopy group $\pi_n(M)$.
- Homotopy groups are Abelian for $n > 1$, i.e $[\alpha] * [\beta] = [\beta] * [\alpha]$.

Summary of important results

- Manifolds M with $\pi_1(M) = 1$ are called *simply-connected*.
- $\pi_n(S^n) = \mathbb{Z}$
  (the integer is known as the *degree* of the map and is related to the number of pre-images)
- $\pi_n(S^d) = 1$ for $1 \leq n < d$
  (contractible, not onto)
- $\pi_{n+1}(S^n) = \mathbb{Z}_2$, for $n \geq 3$, but $\pi_3(S^2) = \mathbb{Z}$ (related to Hopf bundle)
- $\pi_{n+2}(S^2) = \mathbb{Z}_2$ for $n \geq 2$.
  (Homotopy groups of spheres really are complicated!)
- *Spectral sequences* are an important tool:
  Let G be a Lie group with subgroup H then

  $$\cdots \to \pi_n(H) \to \pi_n(G) \to \pi_n(G/H) \to \pi_{n-1}(H) \to \pi_{n-1}(G) \to \pi_{n-1}(G/H) \to \dots$$

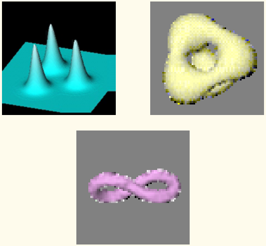
  is a long exact sequence. (example: $G = S^3$, $H = S^1$, $G/H = S^2$)

Homotopy groups and Field Theory

- *Why are these homotopy groups important for field theories?*
- Field configurations are maps $\phi : \mathbb{R}^d \to M$, from flat space to a target space.
- Homotopies of maps occur naturally (e.g. time evolution is continuous and connects different field configurations in the same homotopy class).
- Two scenarios naturally give rise to homotopy groups. Both arise from boundary conditions (due to finite energy).
    1. *One-point compactification:* There is a unique vacuum $v_0 \in M$, namely, $\phi(\mathbf{x}) = v_0$ for $\mathbf{x} \to \infty$. So, we can identify all these points, so that topologically $\mathbb{R}^d \cup \{\infty\} = S^d$. So, we need
    $$\pi_d(M).$$
    2. *Nontrivial maps at infinity:* The vacuum is degenerate and forms a submanifold N of M. Then, in the limit $|\mathbf{x}| \to \infty$ there is a continuous map $\phi|_\infty : S^{d-1}_\infty \to N$. So, we need
    $$\pi_{d-1}(N).$$

Classification of solitons

$\pi_n(S^k)$	ungauged	gauged
$\pi_1(S^1)$	Kinks	**Vortices**
$\pi_2(S^2)$	Baby-Skyrmions, Lumps	Monopoles
$\pi_3(S^3)$	Skyrmions	Instantons
$\pi_3(S^2)$	Hopf Solitons	



Ginzburg-Landau vortices

- The Ginzburg-Landau energy is given by

$$V = \frac{1}{2}\int \left(B^2 + \overline{D_i\phi} D_i\phi + \frac{\lambda}{4}\left(1 - \bar{\phi}\phi\right)^2 \right) \mathrm{d}^2x.$$

  where $\mathbf{x} = (x, y)$.

- This is invariant under

$$\begin{array}{rcl} \phi(\mathbf{x}) & \mapsto & e^{i\alpha(\mathbf{x})}\phi(\mathbf{x}) \\ a_i(\mathbf{x}) & \mapsto & a_i(\mathbf{x}) + \partial_i\alpha(\mathbf{x}), \end{array}$$

  where $e^{i\alpha(\mathbf{x})}$ is a spatially varying phase.

- Here $D_i = \partial_i\phi - ia_i\phi$ is the covariant derivative and

$$B = \partial_1 a_2 - \partial_2 a_1$$

  is the magnetic field.

- The vacuum is $\phi = 1$, $a_i = 0$ and gauge transformations of this.

Topological charge I

- Asymptotically, for finite energy fields, we can fix the gauge so that

  $$\lim_{\rho\to\infty} \phi(\rho,\theta)$$

  exists and varies continuously with θ, where $(x,y) = (\rho\cos\theta, \rho\sin\theta)$.
- Since $|\phi| \to 1$ as $\rho \to \infty$,

  $$\lim_{\rho\to\infty} (\rho,\theta) = e^{i\alpha(\theta)},$$

  where α is a continuous function of θ.
- *Winding number N*: As θ increases from 0 to 2π, $\alpha(\theta)$ increases by $2\pi N$ (ϕ is single valued). N is an arbitrary integer, cannot change under smooth deformations of the field, remains constant in time.
- N is also invariant under smooth gauge transformations.

Topological charge II

- In polar coordinates (ρ, θ)

$$V = \frac{1}{2} \int_0^\infty \int_0^{2\pi} \left(B^2 + \overline{D_\rho \phi} D_\rho \phi + \frac{1}{\rho^2} \overline{D_\theta \phi} D_\theta \phi + \frac{\lambda}{4} \left(1 - \bar{\phi}\phi\right)^2 \right) \rho \, \mathrm{d}\rho \, \mathrm{d}\theta.$$

- By Stokes theorem

$$\int_{\mathbb{R}^2} B \, \mathrm{d}^2 x = \int_0^{2\pi} a_\theta \, \mathrm{d}\theta \Bigg|_{\rho \to \infty}$$

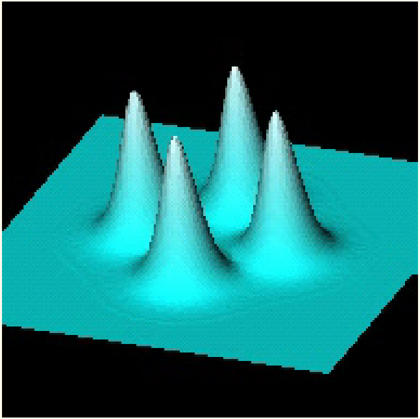
- As $\rho \to \infty$, the covariant derivative $D_\theta \phi = \partial_\theta - i a_\theta \phi$ has to vanish. Since $\phi = e^{i\alpha(\theta)}$ we have $a_\theta = \frac{\mathrm{d}\alpha}{\mathrm{d}\theta}$. Hence

$$\int_{\mathbb{R}^2} B \, \mathrm{d}^2 x = \alpha(2\pi) - \alpha(0) = 2\pi N.$$

so N measures the magnetic flux units in the plane.

Topological charge III

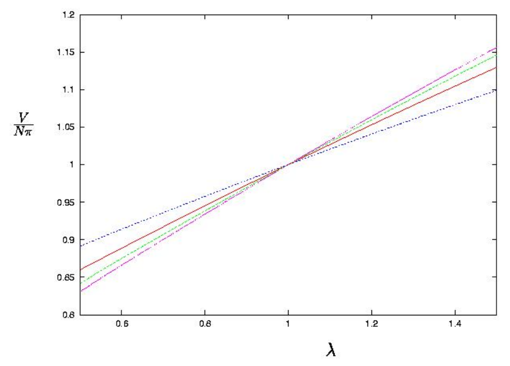
- If ϕ has only isolated zeros, then the number of these (counted with multiplicity) is N.
- A zero of ϕ is said to have multiplicity k, if on a small cirlce enclosing the zero, $-\arg\phi$ increases by $2\pi k$. For simple zeros $k = \pm 1$.



Energy of N Ginzburg-Landau vortices

- Let E_N be the minimal energy V of N vortices.

$\lambda < 1$	$E_N < NE_1$	the vortices attract (Type I)
$\lambda > 1$	$E_N > NE_1$	the vortices repel (Type II)
$\lambda = 1$	$E_N = NE_1$	no forces between static vortices



Vortices at critical coupling $\lambda = 1$

- By "completing the square" V can be written as

$$V = \frac{1}{2}\int\left(\left(B - \frac{1}{2}\left(1 - \bar{\phi}\phi\right)\right)^2 + \left(\overline{D_1\phi + iD_2\phi}\right)\left(D_1\phi + iD_2\phi\right) + B\right)\mathrm{d}^2x.$$

- Recall that

$$\int B \; \mathrm{d}^2x = 2\pi N, \quad \text{so} \quad V \geq \pi N.$$

- Bogomolny equations:

$$\begin{aligned} D_1\phi + iD_2\phi &= 0 \\ B - \frac{1}{2}\left(1 - \bar{\phi}\phi\right) &= 0. \end{aligned}$$

- These equations cannot be solved analytically. However, a lot is known about the solutions.

The Vortex moduli space

- For given topological charge N, the Bogomolny equations have a $2N$ dimensional manifold of static solutions, known as the *moduli space* M_N. (Gauge equivalent solutions are identified.)
- All zeros of ϕ have positive multiplicity (generically there are only simple zeros).
- A solution is completely determined by the locations of these zeros, which can be anywhere. N unordered points in $\mathbb{R}^2$ require $2N$ coordinates.
- There are no *static* forces between vortices for $\lambda = 1$, however, there will be *velocity dependent* forces.

Relativistic vortex dynamics

- The standard relativistic Lagrangian is

$$\mathcal{L} = \frac{1}{2}\overline{D_\mu \phi} D^\mu \phi - \frac{1}{4} f_{\mu\nu} f^{\mu\nu} - \frac{\lambda}{8}\left(1 - \bar{\phi}\phi\right)^2,$$

where $x^\mu = (t, \mathbf{x})$.

- In the following, we will often use complex coordinates $z = x + iy$.
- We can parametrize the moduli space for $\lambda = 1$ in terms of the vortex positions Z_i. Assuming that Z_i are time dependent gives rise to the reduced Lagrangian

$$L_{\mathrm{red.}} = \frac{1}{2}\sum_{r,s=1}^{N}\left(g_{rs}\dot{Z}_r\dot{Z}_s + g_{r\bar{s}}\dot{Z}_r\dot{\bar{Z}}_s + g_{\bar{r}\bar{s}}\dot{\bar{Z}}_r\dot{\bar{Z}}_s\right) - V_{\mathrm{red.}},$$

where

$$V_{\mathrm{red.}} = \frac{\lambda - 1}{8}\int \left(1 - \bar{\phi}\phi\right)^2 \mathrm{d}^2 x.$$

Properties of the moduli space

- Setting $h = \log|\phi|^2$ the Bogomolny equations imply

$$\nabla^2 h + 1 - e^h = 4\pi \sum_{r=1}^{N} \delta^2(z - Z_r).$$

- The δ functions arise because h has logarithmic singularities at the zeros Z_r of ϕ.
- Expanding h around the point Z_r gives

$$h(z,\bar z) = 2\log|z - Z_r| + a_r + \frac{1}{2}\bar b_r(z - Z_r) + \frac{1}{2} b_r(\bar z - \bar Z_r) + \ldots$$

- After a long calculation

$$L_{\mathrm{red.}} = \frac{\pi}{2} \sum_{r,s=1}^{N} \left(\delta_{rs} + 2\frac{\partial b_s}{\partial Z_r} \right) \dot Z_r \dot{\bar Z}_s - V_{\mathrm{red.}}$$

The metric on the moduli space

- The moduli space metric

$$g = \frac{\pi}{2} \sum_{r,s=1}^{N} \left(\delta_{rs} + 2 \frac{\partial b_s}{\partial Z_r} \right) dZ_r d\bar{Z}_s$$

  is Kähler.
- This structure provides a lot of information about the metric, although it is only know implicitly.
- The moduli space approximation captures the dynamics of vortices, in particular right-angle scattering.

First order vortex dymanics

- The Schrödinger-Chern-Simons Lagrangian

$$\begin{aligned}\mathcal{L}_{SCS} &= \frac{i}{2}\left(\overline{\phi}D_0\phi - \phi\overline{D_0\phi}\right) + Ba_0 + e_1a_2 - e_2a_1 - a_0 \\ &\quad -\frac{1}{2}B^2 - \frac{1}{2}\overline{D_i\phi}D_i\phi - \frac{\lambda}{8}\left(1-\overline{\phi}\phi\right)^2,\end{aligned}$$

  is a model for vortex dynamics in superconductors.
- This Lagrangian is gauge invariant and Galilean invariant.
- $\mathcal{L}_{SCS}$ give rise to first order vortex dynamics.
- For λ close to one, we can again use our moduli space M_N to approximate the dynamics of N vortices.

Moduli approximation and the Kähler potential

- Now, the reduced Lagrangian is also first order

  $$L_{\mathrm{red.}} = -\sum_{i=1}^{2N} \mathcal{A}_i(\mathbf{y})\dot{y}_i - V_{\mathrm{red.}}(\mathbf{y}),$$

  where $\mathbf{y}$ are the coordinates on the moduli space and

  $$V_{\mathrm{red.}} = \frac{\lambda - 1}{8} \int \left(1 - \overline{\phi}\phi\right)^2 \mathrm{d}^2 x.$$

- $\mathcal{A}$ is a gauge potential, and $\mathcal{F} = d\mathcal{A}$ the corresponding field strength.
- The equations of motion are

  $$\mathcal{F}_{ij}\dot{y}_j = -\frac{\partial V_{\mathrm{red.}}}{\partial y_i}.$$

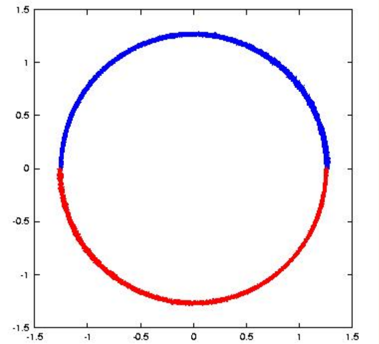
- The field strength $\mathcal{F}$ is

  $$\mathcal{F} = -i\pi \sum_{r,s=1}^{N} \left(\delta_{rs} + 2\frac{\partial b_s}{\partial Z_r}\right) dZ_r \wedge d\bar{Z}_s$$

  which is the Kähler form associated to the metric g on M_N.

Moduli space approximation

- For λ close to 1, two vortices circle around each other anticlockwise.
- Moduli space approximation is in agreement with numerical simulation.



Vortices on various domains

- We can consider physical spaces with a different metric, e.g.

$$ds^2 = dt^2 - \Omega(x,y)(dx^2 + dy^2),$$

  where Ω is a Riemannian metric on a physical space X.

- Again we can "complete the square" and obtain the Bogomolny equations

$$\begin{aligned} D_1\phi + iD_2\phi &= 0 \\ B - \frac{\Omega}{2}\left(1 - \bar{\phi}\phi\right) &= 0, \end{aligned}$$

  where $B = f_{12}$.

- The integral

$$c_1 = \frac{1}{2\pi}\int_X f = \frac{1}{2\pi}\int_X B \, \mathrm{d}^2 x$$

  is an integer. This topological invariant is known as the first *Chern number.*

Compact domains and the Bradlow limit

- We can integrate the second Bogomolny equation over X and obtain

$$2\int_X B\ \mathrm{d}^2x + \int_X |\phi|^2\Omega\ \mathrm{d}^2x = \int_X \Omega\ \mathrm{d}^2x.$$

- If X has a finite area A we obtain

$$4\pi N + \int_X |\phi^2|\Omega\ \mathrm{d}^2x = A.$$

- This gives us the Bradlow limit

$$A \geq 4\pi N$$

  in other words, a vortex needs at least an area of 4π.
- At the Bradlow bound $A = 4\pi N$ both equations can trivially be solved by $\phi = 0$ and $B = \frac{\Omega}{2}$.
- For the torus T^2 the moduli space metric has been calculated as an expansion around the Bradlow limit.

Hyperbolic vortices

- Setting $h = \log|\phi|^2$ we can again derive an equation for h :

$$\nabla^2 h + \Omega - \Omega e^h = 4\pi \sum_{r=1}^{N} \delta^2(z - Z_r).$$

- For hyperbolic space

$$ds^2 = \frac{8}{(1-|z|^2)^2} dz\ d\bar{z}$$

  with $|z| < 1$, the equation can be transformed to Liouville's equation, which is integrable.
- In this case, the moduli space is known explicitly, and

$$\phi = \frac{1-|z|^2}{1-|f|^2} \frac{df}{dz}.$$

  $f(z)$ has the rather simple form

$$f(z) = \prod_{i=1}^{N+1} \left(\frac{z - c_i}{1 - \bar{c}_i z} \right)$$

  where $|c_i| < 1$. The positions of the vortices are the zeros of $\frac{df}{dz}$.

Metric for Hyperbolic vortices

- In hyperbolic space, the metric is

$$g = \frac{\pi}{2} \sum_{r,s=1}^{N} \left(\Omega(Z_r)\delta_{rs} + 2\frac{\partial b_s}{\partial Z_r} \right) dZ_r d\bar{Z}_s$$

  but now we can calculate b_s for special cases.

- The metric for n vortices on a regular polygon with m vortices fixed at the origin is given by

$$ds^2 = \frac{4\pi n^3 |\alpha|^{2n-2} d\alpha \; d\bar{\alpha}}{\left(1-|\alpha|^{2n}\right)^2} \left(1 + \frac{2n\left(1+|\alpha|^{2n}\right)}{\sqrt{(m+1)^2\left(1-|\alpha|^{2n}\right)^2 + 4n^2|\alpha|^{2n}}} \right)$$

  for $n \neq m+1$, and by

$$ds^2 = \frac{12\pi n^3 |\alpha|^{2n-2} d\alpha \; d\bar{\alpha}}{\left(1-|\alpha|^{2n}\right)^2}$$

  for $m+1 = n$. The nontrivial zeros are at $z = \alpha \ \mathrm{e}^{2\pi i k/n}$ for $k = 0, \ldots, n-1$.