1

Noether's Theorem for Smooth, Difference and Finite Element Systems

Elizabeth L. Mansfield
Institute of Mathematics, Statistics and Actuarial Science
University of Kent, Canterbury CT2 7NF UK

Dedicated to the memory of Karin Gatermann *1961–2005*

1.1 Introduction

The question, "Is the long term qualitative behaviour of numerical solutions accurate?" is increasingly being asked. One way of gauging this is to examine the success or otherwise of the numerical code to maintain certain conserved quantities such as energy or potential vorticity. For example, numerical solutions of a conservative system are usually presented together with plots of energy dissipation. But what if the conserved quantity is a less well studied quantity than energy or is not easily measured in the approximate function space? What if there is more than one conserved quantity? Is it possible to construct an integrator that maintains, *a priori*, several laws at once?

Arguably, the most physically important conserved quantities arise via Noether's theorem; the system has an underlying variational principle and a Lie group symmetry leaves the Lagrangian invariant. A Lie group is a group whose elements depend in a smooth way on real or complex parameters. Energy, momentum and potential vorticity, used to track the development of certain weather fronts, are conserved quantities arising from translation in time and space, and fluid particle relabelling respectively. The Lie groups for all three examples act on the base space which is discretised. It is not obvious how to build their automatic conservation into a discretisation, and expressions for the conserved quantities must be known exactly in order to track them.

The study of Lie group symmetries of differential equations is one of the success stories of symbolic computation, (Hereman (1997)). Not only symmetries but integration techniques based on them are now commer-

cially available, (Cheb-Terrab, Duarte and da Mota, (1998)). Moreover, these can usually be obtained without understanding the underlying theory: no human interaction with the software is required. This success is built on the fact that explicit, exact, analytic formulae are known for all the requisite quantities (see for example Olver, (1993)), and the algorithms which are required for the intermediate processing are well understood (Hubert (2000), Hubert (2003), Mansfield and Clarkson (1997), Reid (1991), Reid, Wittkopf and Boulton, (1996)) .

One possibility is to use symbolic methods to study symmetries and conservation laws of discrete systems. One might then calculate intrinsically conserved quantities of existing schemes, but so far this line of research has been less successful for a variety of reasons. The philosophical points of view that are possible for such a theory are still debated, and the computations involved are less tractable than those for smooth systems (Hydon (2000), Levi, Tremblay and Winternitz, (2005), Quispel, Capel and Sahadevan, (1992)).

The key objective of the present article is to examine the idea of making a conservation law an *intrinsic* property of a scheme by building in a symmetry and a discrete variational principle. There are several challenges to this approach. The first is to show how a group action that takes place in a base space that gets discretised is nevertheless still present in some sense. The second is to present a mathematical structure that allows a discrete conservation law to be proven rigorously from the existence of the symmetry.

At the simplest level, the proof of Noether's theorem for smooth systems involves symbolic manipulation of the formulae involved. It is necessary to dig a little deeper to see what might transfer to a discrete setting. The algebraic foundation and the mathematical structures using which Noether's theorem can be proved and elucidated involve the construction of a *variational complex* (see Olver, (1993) and references therein). A complex is an *exact* sequence of maps, that is, the kernel of one map equals the image of the previous map in the sequence. The familiar grad – curl – div sequence is *locally exact*, that is, is exact provided the domain of the functions involved is diffeomorphic to a disc. The variational complex involves the extended sequence of operators, grad – curl – div – Euler-Lagrange – Helmholtz. This extension makes sense if the coefficient functions involve arbitrary dependent variables and their derivatives, as indeed a Lagrangian does. Exactness means, for example, an expression is a divergence if and only if it maps to zero under the Euler–Lagrange operator.

Variational methods for difference systems have been available in the literature for some time (Kupershmidt (1985)). The complete set of proofs showing the difference variational complex is exact were given more recently (Hydon and Mansfield (2004)). There is no tangent structure on a discrete lattice, and so no "top down" construction for the variational complex for difference systems can exist. Yet the formulae involved in the difference version of Noether's theorem are amazingly similar, to the point where a "syntax translation" tells you how to convert one to its counterpart. An independent view and derivation of Noether's theorem for difference systems has been given by Dorodnitsyn (2001).

In the third part of this paper, we discuss how the algebraic arguments transfer to moment-based approximations on an arbitrary triangulation. Classical constructions from algebraic topology, such as simplicial spaces, chains and cochains, boundary and coboundary operators, are needed for this. These ideas are of increasing interest to both physicists and numerical analysts (Chard and Shapiro (2000), Hiptmair (2002), Mattiussi (1997), Schwalm, Moritz, Giona and Schwalm (1999), Tonti (1975)). The interplay of such notions with physical quantities and systems is being explored as a way to ensure that the correct geometry of a problem is encoded in the discretisation.

Our arguments require that the set of moments used fits into an exact scheme as described by Arnold (2002). This means that the various projections to finite dimensional function spaces need to maintain the exactness of the grad – curl – div sequence. Exactness guarantees the conditions for numerical stability given by Brezzi's theorem (Brezzi (1974)), so these ideas have innate meaning for the finite element method quite apart from those presented here. The variational complex for such schemes is detailed in Mansfield and Quispel (2005). Here, we develop those ideas further to investigate Noether's Theorem for finite element approximations.

In Section 1.1, a brief look at Noether's theorem for smooth systems tells the story in a way that the analogies for finite difference and finite element can be easily seen. This is followed by a discussion of the variational complex for difference systems. We define the difference Euler-Lagrange operator, explain how group actions are inherited, and give some examples. Also included is a discussion of how the theory of moving frames can be used to find difference invariants of given Lie group actions. These are used to construct a Lagrangian which *a priori* will have a conservation law corresponding to the given group.

The main result of this paper can be summarised as follows: instead of proving approximate conservation of an exact quantity, we demonstrate the possibility of exact conservation of an associated approximate quantity. The examples are deliberately small and straightforward.

1.2 Brief review of Noether's theorem for smooth systems

Definition 1.2.1 *A* conservation law *for a system of differential equations is a divergence expression which is zero on solutions of the system.*

For example, the heat equation $u_t+(-u_x)_x = 0$ is its own conservation law. To move from the divergence form to the more usual integral form, integrate over an arbitrary domain, assume ∂_t and $\int$ commute, and apply Stokes' Theorem, to obtain,

$$\frac{\partial}{\partial t}\int_\Omega u + (-u_x)]_{\partial\Omega} = 0.$$

In words, this equation reads, "the rate of change of total heat in Ω equals the net of comings and goings of heat across the boundary." The conserved quantity is (usually) that behind the time derivative in the divergence expression.

Noether's theorem provides a conservation law for an Euler–Lagrange system where the Lagrangian is invariant under a Lie group action. The Lagrangian here includes the volume form in the action integral, so we speak of the *Lagrangian form*. Table 1.1 gives the standard names of the conserved quantity for the most common group actions arising in physical applications.

1.2.1 The Euler–Lagrange Equations

The Euler–Lagrange equations are the result of applying a "zero derivative" condition when the dependent variable in a Lagrangian form is varied.

Symmetry	**Conserved Quantity**
$\begin{cases} t^* = t + c \\ \text{translation in time} \end{cases}$	Energy
$\begin{cases} x_i^* = x_i + c \\ \text{translation in space} \end{cases}$	Linear Momenta vector
$\begin{cases} \mathbf{x}^* = \mathcal{R}\mathbf{x} \\ \text{rotation in space} \end{cases}$	Angular Momenta vector
$\begin{cases} a^* = \phi(a,b), b^* = \psi(a,b) \\ \phi_a\psi_b - \phi_b\psi_a \equiv 1 \\ \text{Particle relabelling} \end{cases}$	Potential vorticity

Table 1.1. *The usual examples*

Example 1.2.2 *If the Lagrangian is* $L[u] = \frac{1}{2}\left(u_x^2 + u_{xx}^2\right)\mathrm{d}x$ *then the variation of* $L[u]$ *in the direction* v *is, by definition,*

$$\begin{aligned} \widehat{\mathrm{d}}L[u](v) &= \tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\big|_{\epsilon=0}\, L[u+\epsilon v] \\ &= (u_x v_x + u_{xx} v_{xx})\mathrm{d}x \\ &= (-u_{xx}v + u_{xxxx}v)\mathrm{d}x + \tfrac{\mathrm{D}}{\mathrm{D}x}\left(u_x\mathrm{d}u - 2u_{xx}v_x + \tfrac{\mathrm{D}}{\mathrm{D}x}\left(u_{xx}v\right)\right) \\ &= E(L)v\mathrm{d}x + \tfrac{\mathrm{D}}{\mathrm{D}x}\eta(L,v). \end{aligned}$$

The Euler–Lagrange equation for this Lagrangian is $u_{xxxx} - u_{xx} = 0$.

For the purposes of this article, the way to think of the Euler–Lagrange operator is as $E = \pi \circ \widehat{\mathrm{d}}$, where π projects out the total derivative (total divergence) term. In the case of more than one dependent variable, where each one varies separately, we obtain an equation for each dependent variable. For example,

$$\begin{aligned} \widehat{\mathrm{d}}L[u_1,u_2](v_1,v_2) &= \tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\big|_{\epsilon=0} L[u^1+\epsilon v^1, u^2+\epsilon v^2] \\ &= E^1(L)v^1\mathrm{d}x + E^2(L)v^2\mathrm{d}x + \tfrac{\mathrm{D}}{\mathrm{D}x}\eta(L,v). \end{aligned}$$

The Euler–Lagrange system is then $E^i(L) = 0$, $i = 1, 2$. General formulae, explicit, exact, symbolic, for E^i and $\eta(L,v)$ are known, (Olver (1993)).

1.2.2 Variational Symmetries

Symmetries of differential structures are studied in terms of Lie group actions. A Lie group is one whose elements can be parametrised smoothly

by real (or complex) numbers. (More technically, a Lie group is a differentiable manifold with a group product, such that the multiplication and the inverse maps are smooth functions.) It turns out it is sufficient to study actions of one-parameter subgroups of Lie groups.

Definition 1.2.3 *A subgroup of the Lie group G is called a* one-parameter subgroup *if it is parametrised by $\mathbb{R}$, so that $g(\epsilon) \in G$ for all $\epsilon \in \mathbb{R}$, and*

$$g(\epsilon) \cdot g(\delta) = g(\epsilon + \delta).$$

For example, the set

$$\left\{ \begin{pmatrix} \exp(\epsilon) & 0 \\ 0 & \exp(-\epsilon) \end{pmatrix} \mid \epsilon \in \mathbb{R} \right\}$$

is a one-parameter subgroup of $SL(2, \mathbb{R})$, the special linear group of 2×2 matrices with determinant equal to one.

Definition 1.2.4 *A (right) action of a group G on a space M is a smooth map*

$$G \times M \to M, \qquad (g, z) \mapsto g \cdot z$$

such that

$$g_1 \cdot (g_2 \cdot z) = (g_2 g_1) \cdot z.$$

For a one-parameter subgroup this becomes

$$g(\delta) \cdot (g(\epsilon) \cdot z) = g(\delta + \epsilon) \cdot z.$$

Example 1.2.5 *For the group $G = (\mathbb{R}, +)$, that is, the real numbers under addition, the* projective action *on the plane is given by*

$$\epsilon \cdot x = x^* = \frac{x}{1 - \epsilon x}, \qquad \epsilon \cdot u = u^*(x^*) = \frac{u(x)}{1 - \epsilon x}. \tag{1.1}$$

This is actually only a *local* action since ϵ is restricted to a neighbourhood of $0 \in \mathbb{R}$, where the neighbourhood depends on x. We demonstrate the group action property:

$$\delta \cdot (\epsilon \cdot x) = \delta \cdot \left(\frac{x}{1 - \epsilon x} \right) = \frac{\dfrac{x}{1 - \delta x}}{1 - \epsilon \dfrac{x}{1 - \delta x}} = \frac{x}{1 - (\epsilon + \delta) x} = (\epsilon + \delta) \cdot x.$$

For actions on $\mathcal{X} \times \mathcal{U}$ where $\mathcal{X}$ is the space of independent variables and $\mathcal{U}$ the space of dependent variables, then an action is induced on the

associated jet bundle. This is called the *prolongation* action and is obtained using the chain rule of undergraduate calculus. Thus, continuing Example 1.2.5,

$$\epsilon \cdot u_x = u^*_{x^*} = \frac{\partial u^*(x^*)}{\partial x} \Big/ \frac{\partial x^*}{\partial x} = \frac{u_x}{(1-\epsilon x)^2}$$

and checking this indeed gives a group action,

$$\delta \cdot (\epsilon \cdot u_x) = \frac{\delta \cdot u_x}{(1-\epsilon(\delta \cdot x))^2} = \frac{u_x}{(1-(\delta+\epsilon)x)^2}.$$

Given a prolongation action, we then have an induced action on the integral of the Lagrangian form, given by

$$\begin{aligned} \epsilon \cdot \textstyle\int_\Omega L(x,u,u_x,\ldots)\,\mathrm{d}x \quad &:= \quad \textstyle\int_{\epsilon\cdot\Omega} L(\epsilon\cdot x, \epsilon\cdot u, \epsilon\cdot u_x, \cdots)\,\mathrm{d}\epsilon\cdot x \\ &= \quad \textstyle\int_\Omega L(\epsilon\cdot x, \epsilon\cdot u, \epsilon\cdot u_x, \cdots)\dfrac{\mathrm{d}\epsilon\cdot x}{\mathrm{d}x}\,\mathrm{d}x \end{aligned} \tag{1.2}$$

where the first line is the definition of a group action on an integral, and the second follows by regarding the group action as a change of variable, back to the original domain. If the Lagrangian is invariant under this group action for arbitrary Ω, then by standard arguments (involving the Hilbert space L^2),

$$L(x,u,u_x,\ldots) = L(\epsilon\cdot x, \epsilon\cdot u, \epsilon\cdot u_x, \cdots)\frac{\mathrm{d}\epsilon\cdot x}{\mathrm{d}x}.$$

for all ϵ.

Definition 1.2.6 *The infinitesimal action corresponding to that of a one-parameter group with parameter ϵ is obtained by applying $\dfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}$ to the transformed variables.*

Continuing Example 1.2.5, we have

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \epsilon\cdot x = x^2, \qquad \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \epsilon\cdot u = xu, \qquad \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \epsilon\cdot u_x = 2xu_x.$$

If i indexes the independent variables and α indexes the dependent variables, we denote the infinitesimal action on these by

$$\phi^\alpha = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \epsilon\cdot u^\alpha, \quad \xi_i = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \epsilon\cdot x_i. \tag{1.3}$$

Definition 1.2.7 *With ϕ^α and ξ_i as defined in (1.3), the* characteristic *of the group action is the vector $Q = (Q^\alpha)$, with*

$$Q^\alpha = \phi^\alpha - \sum_i \xi_i u^\alpha_{x_i}.$$

We can now state **Noether's Theorem**.

Theorem 1.2.8 *If Q^α are the characteristics of a variational symmetry of a Lagrangian form, then*

$$Q \cdot E(L) = \sum_\alpha Q^\alpha E^\alpha(L) = \mathrm{Div}(\mathcal{A}(Q, L))$$

where precise expressions (symbolic, exact, analytic) for $\mathcal{A}(L, Q)$ are known (Olver (1993), Proposition 5.74).

In words, given a symmetry of a Lagrangian, there is a divergence expression, $\mathrm{Div}(\mathcal{A}(L, Q))$ which is zero on solutions of the Euler–Lagrange system, $E^\alpha(L) = 0$.

On the simplest level, the proof involves a manipulation of the expressions involved. In order to translate the theorem to a discrete setting, we need to look at the *algebraic* underpinning of the proof. This consists of the variational complex which we now briefly describe. Full details may be found in (Olver (1993)).

1.2.3 The variational complex

The variational complex based on a p-dimensional space is constructed from the commutative diagram,

$$\begin{array}{ccccccccc} \xrightarrow{\mathbf{D}} & \Lambda^{p-1} & \xrightarrow{\mathbf{D}} & \Lambda^p & \xrightarrow{\widehat{\mathrm{d}}} & \widehat{\Lambda}_1 & \xrightarrow{\widehat{\mathrm{d}}} & \widehat{\Lambda}_2 & \xrightarrow{\widehat{\mathrm{d}}} \\ & & & & & \downarrow \pi & & \downarrow \pi & \\ & & & & & {\Lambda^1}_* & \xrightarrow{\delta} & {\Lambda^2}_* & \xrightarrow{\delta} \end{array} \tag{1.4}$$

Brief description of the components of (1.4):
The spaces Λ^k on the left of the diagram (1.4) are k-forms in the independent variables, but where the coefficients may depend, in a smooth way, on a finite number of dependent variables and their derivatives. The map $\mathbf{D}$ is the *total* exterior derivative. For example, in two dimensions, $\mathbf{D}(u_x \mathrm{d}y) = u_{xx} \mathrm{d}x\mathrm{d}y$. The spaces $\widehat{\Lambda}^j$ are the so-called *vertical* k-forms, that is, forms in the dependent variables and their derivatives, multiplied by the volume form on the base space. For example, in a two dimensional

space, $x^2 u_x u_y \mathrm{d}u\mathrm{d}u_x\mathrm{d}x\mathrm{d}y \in \widehat{\Lambda}^2$. The map $\widehat{\mathrm{d}}$ is the exterior derivatives in the vertical direction. Thus, $\widehat{\mathrm{d}}(xyu_x^2\mathrm{d}x\mathrm{d}y) = 2xyu_x\mathrm{d}u_x\mathrm{d}x\mathrm{d}y$.

The first step in the calculation of the Euler–Lagrange operator in Example 1.2.2 is indeed the map $\widehat{\mathrm{d}}$. Using the exterior form notation, the calculation becomes

$$\begin{aligned}
\widehat{\mathrm{d}}(L\mathrm{d}x) &= \widehat{\mathrm{d}}\left(\tfrac{1}{2}\left(u_x^2 + u_{xx}^2\right)\mathrm{d}x\right) \\
&= (u_x\mathrm{d}u_x + u_{xx}\mathrm{d}u_{xx})\mathrm{d}x \\
&= (-u_{xx}\mathrm{d}u + u_{xxxx}\mathrm{d}u)\mathrm{d}x \\
&\qquad + \tfrac{\mathrm{D}}{\mathrm{D}x}\left(u_x\mathrm{d}u - 2u_{xx}\mathrm{d}u_x + \tfrac{\mathrm{D}}{\mathrm{D}x}\left(u_{xx}\mathrm{d}u\right)\right) \\
&= E(L)\mathrm{d}u\mathrm{d}x + \tfrac{\mathrm{D}}{\mathrm{D}x}\eta(L).
\end{aligned}$$

As is seen in this example, the "integration by parts" step uses an action of the operator $\mathrm{D}/\mathrm{D}x$ on the forms, for example

$$\frac{\mathrm{D}}{\mathrm{D}x}(u^2\mathrm{d}u_x) = 2u\mathrm{d}u_x + u^2\mathrm{d}u_{xx}$$

and so forth. This generalises to higher dimensions, so that there is an action of the total divergence operator on the $\widehat{\Lambda}^k$.

The spaces ${\Lambda^k}_*$ are defined as equivalence classes of vertical forms; two forms are equivalent if they differ by a total divergence. The map d_* is then the maps $\widehat{\mathrm{d}}$ as induced on these classes, while the maps π are the projection maps. The Euler–Lagrange operator is then $\widehat{\mathrm{d}} \circ \pi$.

Definition 1.2.9 *The variational complex, given here for a p-dimensional base space, is*

$$\cdots \xrightarrow{\mathbf{D}} \Lambda^{p-1} \xrightarrow{\mathbf{D}} \Lambda^p \xrightarrow{E} {\Lambda^1}_* \xrightarrow{\mathrm{d}_*} {\Lambda^2}_* \xrightarrow{\mathrm{d}_*} \cdots \tag{1.5}$$

Note that the map d_* is denoted by δ in (Olver (1993)). We reserve the notation δ for the simplicial coboundary map needed in section 1.3.

Theorem 1.2.10 *(Olver (1993)) The complex (1.5) is exact. That is, the image of one map equals the kernel of the next.*

Thus, if $E(L) = 0$ then L is necessarily in the image of $\mathbf{D}$. Since $\mathbf{D}$ on Λ^{p-1} is essentially the total divergence operator, this means that E is zero, and only zero on, total divergences. The proof of this result is constructive, that is, formulae for the pre-images are known. These formulae are given in terms of homotopy operators which can be used, at least in principle, in ansatz methods for finding conservation laws not necessarily arising from Noether's Theorem. In practice, more direct

methods are often used (Wolf (2000), Wolf (2003), Wolf, Brand and Mohammadzadeh, (1999)).

The infinitesimal form of a Lie group action induces an action on forms. To describe this, we make the following definitions.

Definition 1.2.11 *Given the characteristic of an action $Q = (Q^\alpha)$ given in Definition 1.2.7, we define the* characteristic vector

$$\mathbf{v}_Q = \sum_{\alpha,K} \mathrm{D}^K(Q^\alpha)\frac{\partial}{\partial u^\alpha_K}$$

where α indexes the dependent variables and K is a multi-index of differentiation.

The inner product of a vector with a form is given by

$$\frac{\partial}{\partial u^\alpha_K} \lrcorner\, \mathrm{d}u^\beta_J = \delta^\alpha_\beta \delta^K_J, \qquad \frac{\partial}{\partial u^\alpha_K} \lrcorner\, \mathrm{d}x_i = 0$$

where δ is the Kronecker delta, and acts on products as a signed derivation. Thus, for example,

$$\mathbf{v} \lrcorner\, f(u)\mathrm{d}u_x\mathrm{d}u_{xx} = f(u)\frac{\mathrm{D}Q}{\mathrm{D}x}\mathrm{d}u_{xx} - f(u)\frac{\mathrm{D}^2Q}{\mathrm{D}x^2}\mathrm{d}u_x.$$

Noether's theorem is obtained by considering the map $\mathbf{v}_Q \lrcorner\, \circ\, \widehat{\mathrm{d}}$ and hence $\mathbf{v}_Q \lrcorner\, \circ E$;

$$\xrightarrow{\mathbf{D}} \Lambda^p \underset{\mathbf{v}_Q \lrcorner}{\overset{E}{\rightleftarrows}} {\Lambda^1}_* \xrightarrow{\mathrm{d}_*} \tag{1.6}$$

It is straightforward to show that the induced infinitesimal action of a Lie group on a Lagrangian form has the formula,

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \epsilon \cdot L[u] = \mathbf{v}_Q \lrcorner\, \widehat{\mathrm{d}}L[u] + \mathrm{Div}(L\boldsymbol{\xi}) \tag{1.7}$$

where $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_p)$.

If the Lagrangian is invariant, the left-hand side of (1.7) will be zero. Since $\widehat{\mathrm{d}}L[u] = \sum E^\alpha(L)\mathrm{d}u^\alpha\mathrm{d}\mathbf{x} + \mathrm{Div}(\eta(L))$ and $\mathbf{v}_Q$ has no $\partial/\partial x_i$ terms, so that Div and $\mathbf{v}_Q \lrcorner\,$ commute, Noether's theorem follows.

Example 1.2.12 *Consider the Lagrangian, $L[u] = \frac{1}{2}\left(\frac{u_x}{u}\right)^2 \mathrm{d}x$, which is invariant under both translation in x and scaling in u. The associated*

Euler–Lagrange equation is

$$E(L) = \frac{u_x^2}{u^3} - \frac{u_{xx}}{u^2}.$$

For the translation action, $Q = -u_x$ *since* $\phi = 0$ *and* $\xi = 1$. *And indeed,*

$$-u_x E(L) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u_x^2}{u^2}\right).$$

For the scaling action, $Q = u$, *as* $\phi = u$ *and* $\xi = 0$, *and so*

$$uE(L) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u_x}{u}\right).$$

A more significant example can be found in (Bila, Mansfield and Clarkson, (2005)) where conservation laws arising from symmetries of a meteorological model are classified.

In summary, the algebraic part of the proof of Noether's theorem involves a variational complex and an infinitesimal group action. Emulating the algebraic pattern, rather than the analysis, is the key to success for the construction and proof of the discrete Noether's Theorems.

We next look at the translation of these concepts for difference systems.

1.3 Difference Systems

We will consider a difference Lagrangian $L[u_n^\alpha]$ to be a smooth function of a finite number of difference variables and their shifts. Such difference Lagrangians may result from a discretisation of a smooth Lagrangian, but not necessarily. Since there exist inherently discrete systems with perhaps no continuum limit, we limit the types of calculations we perform here strictly to those operations pertinent to such systems.

We regard the lattice coordinates $\mathbf{n} = (n^1, \dots, n^p) \in \mathbb{Z}^p$ as the independent variables. The dependent variables $\mathbf{u_n} = (u^1_{\mathbf{n}}, \dots, u^q_{\mathbf{n}})$ are assumed to vary continuously and to take values in $\mathbb{R}$. Let $\mathbf{1}_k$ be the p-tuple whose only nonzero entry is in the k^{th} place; this entry is 1. Then the k^{th} shift map acts as

$$S_k : \mathbf{n} \mapsto \mathbf{n} + \mathbf{1}_k, \qquad S_k : f(\mathbf{n}) \mapsto f(\mathbf{n} + \mathbf{1}_k) \qquad S_k : u^\beta_{\mathbf{n}} \mapsto u^\beta_{\mathbf{n}+\mathbf{1}_k},$$
$$S_k : f(\mathbf{n}, \dots, u^\beta_{\mathbf{n}+\mathbf{m}}, \dots) \mapsto f(\mathbf{n} + \mathbf{1}_k, \dots, u^\beta_{\mathbf{n}+\mathbf{m}+\mathbf{1}_k}, \dots)$$

where f is a smooth function of its arguments. Note that the shift maps

commute (i.e. $S_k S_j = S_j S_k$), We write the composite of shifts using multi-index notation as

$$S^{\mathbf{m}} = S_1^{m_1} \dots S_p^{m_p} \tag{1.8}$$

so that, for example, $u^{\beta}_{\mathbf{n}+\mathbf{m}} = S^{\mathbf{m}} u^{\beta}_{\mathbf{n}}$.

Definition 1.3.1 *A function $F[u^{\alpha}_n]$ is said to be a* total difference *if there is a vector $(A_1[u^{\alpha}_n], \cdots A_p[u^{\alpha}_n])$ such that*

$$F = (S_1 - \mathrm{id})A_1 + \cdots + (S_p - \mathrm{id})A_p.$$

Definition 1.3.2 *A* difference conservation law *for a difference system is a total difference which is zero on solutions.*

Example 1.3.3 *The standard discretisation of the heat equation,*

$$u_{n,m+1} - u_{n,m} = u_{n+1,m} - 2u_{n,m} + u_{n-1,m}$$

is a difference conservation law for itself, since it can be written

$$(S_1 - \mathrm{id})[(S_1 - \mathrm{id})(-u_{n-1,m})] + (S_2 - \mathrm{id})u_{n,m} = 0.$$

Just as an integral of a total divergence depends only on the boundary data, so does the sum over a lattice domain of a total difference.

1.3.1 The difference Euler–Lagrange operator

As with smooth systems, the *difference* Euler–Lagrange equations result from a "zero derivative" condition when a difference Lagrangian is varied with respect to its variables. The "integration by parts" step of the calculation is replaced by, in one dimension,

$$\sum (Sf)_n g_n = \sum f_n (S^{-1}g)_n + (S - \mathrm{id}) \sum (f_n (S^{-1}g)_n).$$

Example 1.3.4

$$\begin{aligned}
\widehat{\mathrm{d}}(L_n) &= \widehat{\mathrm{d}}\left(\tfrac{1}{2}u_n^2 + u_n u_{n+1}\right) \\
&= (u_n \mathrm{d}u_n + u_{n+1}\mathrm{d}u_n + u_n \mathrm{d}u_{n+1}) \\
&= (u_n + u_{n+1} + u_{n-1})\mathrm{d}u_n + (S - \mathrm{id})(u_n \mathrm{d}u_{n+1}) \\
&= E(L_n)\mathrm{d}u_n + (S - \mathrm{id})(\eta(L_n)).
\end{aligned}$$

General formulae, (explicit, exact, symbolic), for E and $\eta(L_n)$ are known (Hydon and Mansfield (2004)).

As for the smooth case, we define the difference Euler–Lagrange operator to be $E = \pi \circ \widehat{\mathrm{d}}$, where π projects out the total difference term. If there is more than one dependent variable, we obtain one equation for each dependent variable, for example in one dimension,

$$\widehat{\mathrm{d}}(L_n[u,v]) = E^u(L_n)\mathrm{d}u_n + E^v(L_n)\mathrm{d}v_n + (S-\mathrm{id})(\eta(L_n)).$$

1.3.2 Difference variational symmetries

If the difference equation arises as a discretisation of a smooth system, where there is a group action on the base space, then we can treat the mesh variables $x_{\mathbf{n}}$ as *dependent* variables (recall the independent variables are now the integer lattice co-ordinates), see Example 1.3.7 below. The induced group action will satisfy the property that the group action commutes with shift:

$$\epsilon \cdot S^j(u_n) = \epsilon \cdot u_{n+j} = S^j \epsilon \cdot u_n$$

for all j. For example,

$$\epsilon \cdot u_n = \frac{u_n}{1-\epsilon x_n} \qquad \text{implies} \qquad \epsilon \cdot u_{n+j} = \frac{u_{n+j}}{1-\epsilon x_{n+j}}.$$

We will assume this property for any group action on a difference system, not just those arising from discretisations.

The symmetry condition is that $L[u_n^\alpha]$ is an invariant function,

$$L_{\mathbf{n}}(u_{\mathbf{n}}^{\alpha_1}, \cdots, u_{\mathbf{n+k}}^{\alpha_\ell}) = L_n(\epsilon \cdot u_{\mathbf{n}}^{\alpha_1}, \cdots, \epsilon \cdot u_{\mathbf{n+k}}^{\alpha_\ell}). \tag{1.9}$$

Defining the characteristics of the symmetry to be

$$Q_{\mathbf{n}}^\alpha = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} u_{\mathbf{n}}^{\alpha *}, \tag{1.10}$$

and applying

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}$$

to both sides of (1.9) yields

$$0 = \sum_k \frac{\partial L_{\mathbf{n}}}{\partial u_{\mathbf{n+k}}^\alpha} Q_{\mathbf{n+k}}^\alpha \tag{1.11}$$

Since by our assumption,

$$Q_{\mathbf{n+k}}^\alpha = S^k(Q_{\mathbf{n}}^\alpha),$$

equation (1.11) can be written as

$$0 = X_Q \lrcorner \widehat{\mathrm{d}} L_{\mathbf{n}}$$

where

$$X_Q = \sum_{\alpha,\,\mathbf{j}} S^{\mathbf{j}}(Q^\alpha_{\mathbf{n}}) \frac{\partial}{\partial u^\alpha_{\mathbf{n}+\mathbf{j}}}.$$

Theorem 1.3.5 The difference Noether's theorem. *If the symmetry condition (1.9) holds, then with the characteristics of the symmetry defined in (1.10),*

$$Q \cdot E(L_{\mathbf{n}}) = \sum_j (S_j - \mathrm{id})(\mathcal{A}^j_{\mathbf{n}}(Q_{\mathbf{n}}, L_{\mathbf{n}})).$$

Thus a symmetry yields a total difference expression which is zero on solutions of the difference Euler–Lagrange system. Explicit formulae for $\mathcal{A}_{\mathbf{n}}(Q_{\mathbf{n}}, L_{\mathbf{n}})$ are known (Hereman, Colagrosso, Sayers, Ringler, Deconinck, Nivala and Hickman, (2005), Hereman, Sanders, Sayers and Wang, (2005), Hydon and Mansfield (2004)). As for the smooth system, these quantities are defined in terms of homotopy operators which may be used to obtain conservation laws, not necessarily arising from Noether's theorem, in ansatz-based methods.

The similarity of the formulae to those of the smooth case is striking, particularly when the formulae for $\mathcal{A}_{\mathbf{n}}(Q_{\mathbf{n}}, L_{\mathbf{n}}))$ and $\mathcal{A}(Q, L))$ are compared. In fact, the algebraic underpinning of the difference Noether's theorem matches that of the smooth. One can build a diagram in complete analogy to (1.4), and the locally exact variational complex for difference systems is

$$\xrightarrow{\Delta} \mathbf{Ex}^{p-1} \xrightarrow{\Delta} \mathbf{Ex}^{p} \xrightarrow{E} \Lambda^1_* \xrightarrow{\mathrm{d}_*} \Lambda^2_* \xrightarrow{\mathrm{d}_*}$$

where $\mathbf{Ex}^n$ is a difference analogue of Λ^n and Λ^j_* are j-forms in the difference dependent variables and their shifts, modulo total differences. The diagram corresponding to (1.6) is

$$\xrightarrow{\Delta} \mathbf{Ex}^{p} \underset{\mathbf{X}_Q \lrcorner}{\overset{E}{\rightleftarrows}} {\Lambda^1}_* \xrightarrow{\mathrm{d}_*} \tag{1.12}$$

Note that the map d_* is denoted by δ in (Hydon and Mansfield (2004)). We reserve the notation δ for the simplicial coboundary map needed in section 1.3.

Remark 1.3.6 *The difference Noether's theorem is independent of any continuum limit. This is important since there are difference systems with multiple limits, or even no continuum limits at all. In cases where the difference system does have a continuum limit, it is interesting to note that in the examples studied, the Euler–Lagrange system and the conservation law also have continuum limits, and indeed limit to their corresponding quantities, but no proof of a general result is known.*

Example 1.3.7 *This elementary example is taken from the Introduction of Lee (1987), and concerns a difference model for the Lagrangian,* $\int(\frac{1}{2}\dot{x}^2 - V(x))\,\mathrm{d}t$. *Define*

$$\bar{V}(n) = \frac{1}{x_n - x_{n-1}} \int_{x_{n-1}}^{x_n} V(x)\,\mathrm{d}x$$

and take

$$L_n = \left[\frac{1}{2}\left(\frac{x_n - x_{n-1}}{t_n - t_{n-1}}\right)^2 - \bar{V}(n)\right](t_n - t_{n-1}).$$

The group action is translation in time, $t_n^* = t_n + \epsilon$, *with* x_n *invariant. The conserved quantity is thus "energy". Now,* $Q_n^t = 1$ *for all* n, *and* $Q_n^x = 0$. *The Euler–Lagrange equation for the* t_n, *viewed as a dependent variable, is*

$$0 \quad = \quad E^t(L_n) \quad = \quad \tfrac{\partial}{\partial t_n} L_n + S\left(\tfrac{\partial}{\partial t_{n-1}} L_n\right)$$

and since L_n *is a function of* $(t_n - t_{n-1})$,

$$0 = E^t(L_n) = (S - \mathrm{id})\left(\frac{\partial}{\partial t_n} L_n\right)$$

verifying the difference Noether Theorem in this case. The first integral (conservation law) is thus

$$\frac{1}{2}\left(\frac{x_n - x_{n-1}}{t_n - t_{n-1}}\right)^2 + \bar{V}(n) = c.$$

Note that the energy in the smooth case is

$$\frac{1}{2}\dot{x}^2 + V$$

showing the continuum limit of the energy for the difference system is the energy for the smooth system.

Remark 1.3.8 *The Euler–Lagrange equations for the mesh variables could well be regarded as an equation for a moving mesh. It may be appropriate to add terms to the difference Lagrangian that keep the mesh from collapsing or folding.*

1.3.3 Building in a conservation law to a difference variational system

If we know the group action for a particular conservation law, we can "design in" that conservation law into a discretisation by taking a Lagrangian composed of invariants. The Fels and Olver formulation of moving frames (Fels and Olver (1998), Fels and Olver (1999)) is particularly helpful here. A sample theorem concerning difference rotation invariants on $\mathbb{Z}^2$ follows. Consider the action,

$$\epsilon \cdot \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}. \tag{1.13}$$

Theorem 1.3.9 *Let (x_n, y_n), (x_m, y_m) be two points in the plane. Then*

$$I_{n,m} = x_n y_n + x_m y_m, \quad J_{n,m} = x_n y_m - x_m y_n$$

generate the invariants under the action (1.13); any planar rotation difference invariant is a function of these.

Example 1.3.10 *We consider a difference Lagrangian which is invariant under the action (1.13). Suppose*

$$L_n = \frac{1}{2} J_{n,n+1}^2 = \frac{1}{2}(x_n y_{n+1} - x_{n+1} y_n)^2.$$

Then the Euler–Lagrange equations are

$$\begin{aligned} E_n^x &= J_{n,n+1} y_{n+1} - J_{n-1,n} y_{n-1}, \\ E_n^y &= -J_{n,n+1} x_{n+1} + J_{n-1,n} x_{n-1}. \end{aligned}$$

Now, $Q_n = (Q_n^x, Q_x^y) = (-y_n, x_n) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\big|_{\epsilon=0}(x_n^*, y_n^*)$ and thus

$$\begin{aligned} Q_n \cdot E_n &= J_{n,n+1}(-y_n y_{n+1} - x_n x_{n+1}) \\ &\quad + J_{n-1,n}(y_n y_{n-1} + x_n x_{n-1}) \\ &= -J_{n,n+1} I_{n,n+1} + J_{n-1,n} I_{n-1,n} \\ &= -(S - \mathrm{id})(J_{n-1,n} I_{n-1,n}) \end{aligned}$$

gives the conserved quantity. Since the group action is a rotation, the conserved quantity is "angular momentum". Note that $I_{n,m} = I_{m,n}$ and $J_{n,m} = -J_{m,n}$.

Knowing the invariants is actually only half the battle, if you also require that the difference Lagrangian has a particular continuum limit. For one-dimensional systems, the theory of multispace can be used to obtain invariance under a given group action and a given limit simultaneously, see Olver (2001), Mansfield and Hydon (2001).

1.4 Finite Element systems

In obtaining a Noether's theorem for finite element approximations, we base our discussion on the variational complex developed in Mansfield and Quispel, (2005). This in turn, is based on the discussion of numerically stable finite element approximations given in Arnold, (2002). We first look at a simple one-dimensional example. The analogies with the finite difference case here are sufficiently strong that we can obtain immediate results. We then discuss the higher-dimensional case.

1.4.1 The one dimensional case

We give an example of a system of moments that fit a commutative diagram and show how the Euler–Lagrange equations are derived. Let the "triangulation" of $\mathbb{R}$ be given by $\ldots x_{n-1}, x_n, x_{n+1}, \ldots$. We choose moment-based approximations for 0-forms (functions), and 1-forms so that the following diagram is commutative in each (x_n, x_{n+1});

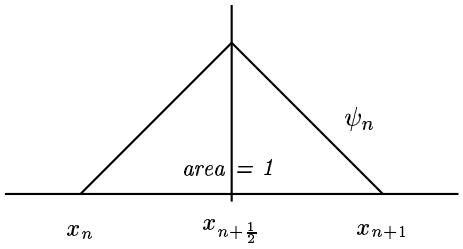
$$\begin{array}{ccccccc} 0 \longrightarrow \mathbb{R} & \longrightarrow & \Lambda^0 & \xrightarrow{\mathrm{d}} & \Lambda^1 & \longrightarrow 0 \\ & & \Pi_0 \downarrow & & \Pi_1 \downarrow & & \\ 0 \longrightarrow \mathbb{R} & \longrightarrow & \mathcal{F}_0 & \xrightarrow{\mathrm{d}} & \mathcal{F}_1 & \longrightarrow 0 \end{array} \tag{1.14}$$

The maps Π_i are projections to piecewise defined forms.

Example 1.4.1 *In this example, the piecewise projection of 1-forms is*

$$f(x)\mathrm{d}x|_{(x_n, x_{n+1})} \mapsto \left(\int_{x_n}^{x_{n+1}} f(x)\psi_n(x)\mathrm{d}x \right) \mathrm{d}x$$

where ψ_n is given diagrammatically as



Here $x_{n+\frac{1}{2}}$ is any intermediate point, and $\psi_n(x_{n+\frac{1}{2}})$ is chosen so that the integral $\int_{x_n}^{x_{n+1}} \psi_n = 1$. The moments used to approximate functions are

$$\alpha_n = \frac{1}{x_{n+\frac{1}{2}} - x_n} \int_{x_n}^{x_{n+\frac{1}{2}}} u(x)\,\mathrm{d}x, \quad \beta_n = \frac{1}{x_{n+1} - x_{n+\frac{1}{2}}} \int_{x_{n+\frac{1}{2}}}^{x_{n+1}} u(x)\,\mathrm{d}x.$$

Commutativity means that

$$\Pi_1(u_x \mathrm{d}x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\Pi_0(u)\right)\mathrm{d}x, \tag{1.15}$$

while the projection property is that

$$\Pi_i \circ \Pi_i = \Pi_i.$$

So, we take the projection of $u|_{(x_n, x_{n+1})}$ using α_n, β_n to be

$$\begin{aligned} u \quad \mapsto \quad & 2\frac{\beta_n - \alpha_n}{x_{n+1} - x_n} x + \left(\frac{x_{n+\frac{1}{2}} + x_{n+1}}{x_{n+1} - x_n}\right)\alpha_n \\ & - \left(\frac{x_{n+\frac{1}{2}} + x_n}{x_{n+1} - x_n}\right)\beta_n \end{aligned}$$

The moments α_n and β_n are not unrelated, however. The formulae are the same, only the domains differ. In effect, $\beta_n = \alpha_{n+\frac{1}{2}}$. So, we can define a shift map S so that

$$\begin{aligned} S(n) &= n + \tfrac{1}{2}, \\ S(\alpha_n) &= \beta_n, \\ S(\beta_n) &= \alpha_{n+1}. \end{aligned}$$

We consider the simplest Lagrangian $\mathcal{L} = \int \frac{1}{2} u_x^2 \,\mathrm{d}x$ which projects to

$$\Pi(\mathcal{L}) = \sum_n \int_{x_n}^{x_{n+1}} \frac{1}{2}\Pi(u)_x^2 \,\mathrm{d}x = \sum_n 2\frac{(\beta_n - \alpha_n)^2}{x_{n+1} - x_n} = \sum L_n.$$

Then

$$\begin{aligned}\hat{\mathrm{d}}L_n &= 4\tfrac{\beta_n-\alpha_n}{x_{n+1}-x_n}(\mathrm{d}\beta_n-\mathrm{d}\alpha_n)\\ &= 4\tfrac{S(\alpha_n)-\alpha_n}{x_{n+1}-x_n}(\mathrm{d}S(\alpha_n)-\mathrm{d}\alpha_n)\\ &= 4\left(S^{-1}\left(\tfrac{S(\alpha_n)-\alpha_n}{x_{n+1}-x_n}\right)-\tfrac{S(\alpha_n)-\alpha_n}{x_{n+1}-x_n}\right)\mathrm{d}\alpha_n+(S-\mathrm{id})(\textit{something}).\end{aligned}$$

The discrete Euler–Lagrange equation is the coefficient of $\mathrm{d}\alpha_n$. *After "integration", and setting* $\beta_n=\alpha_{n+\frac{1}{2}}$,

$$\frac{\alpha_{n+\frac{1}{2}}-\alpha_n}{x_{n+1}-x_n}\equiv c$$

which has the correct continuum limit.

We note that usually the approximation of functions is chosen so that the result is still continuous. This is an additional requirement that our calculations don't seem to need.

The main conjecture is that provided the system of moments used to project the forms fits the commutative diagram (1.14), then an Euler–Lagrange system, in the form of a recurrence system and having the correct continuum limit, can be derived (Mansfield and Quispel (2005)). As earlier, this will be a zero derivative condition obtained when the projected Lagrangian is varied with respect to the independent moments, modulo the analogue of a total difference.

We will show in Section 1.4.3 how a group action acting on dependent and independent variables induces an action on moments given as integrals.

1.4.2 The higher-dimensional case

We give the three-dimensional case; there are no significant changes for higher (or lower!) dimensions.

Given a system of moments and sundry other data, also known as degrees of freedom, we require that these yield projection operators such that the diagram (written here for three-dimensional space) commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R} & \longrightarrow \Lambda^0 & \longrightarrow \Lambda^1 & \longrightarrow \Lambda^2 & \longrightarrow \Lambda^3 & \longrightarrow & 0 \\ & & & \Pi_0\downarrow & \Pi_1\downarrow & \Pi_2\downarrow & \Pi_3\downarrow & & \\ 0 & \longrightarrow & \mathbb{R} & \longrightarrow \mathcal{F}^0 & \longrightarrow \mathcal{F}^1 & \longrightarrow \mathcal{F}^2 & \longrightarrow \mathcal{F}^3 & \longrightarrow & 0 \end{array} \tag{1.16}$$

all relative to some triangulation.

In general, a Lagrangian is composed of wedge products of 1-, 2- , ... p- forms. In Arnold (2002) it is argued that if the approximation of an n-form is taken to be its projection in $\mathcal{F}^n$, then commutativity implies conditions for Brezzi's theorem (Brezzi (1974)), guaranteeing numerical stability, will hold.

Thus a *finite element Lagrangian* is built up of wedge products of forms in $\mathcal{F}_0, \mathcal{F}_1, \ldots \mathcal{F}_{p-1}, \mathcal{F}_p$, with *unevaluated degrees of freedom*. Call the space of such products, $\bar{\mathcal{F}}_p$. In each top-dimensional (p-dimensional) simplex, denoted τ, integrate to get

$$L = \sum_{\tau} L_\tau(\alpha_\tau^1, \cdots \alpha_\tau^p)$$

where α_τ^j is the j^{th} degree of freedom in τ. Note that L can also depend on mesh data $x_{\mathbf{n}}$. We can now take the finite element vertical exterior derivative, $\hat{\mathrm{d}}$, to be the variation with respect to the α_τ^j.

There will be analogues of the shift maps that take moments defined on one simplex to moments defined on nearby simplexes.

The analogue of total divergence or total difference is the *coboundary* concept from simplicial algebraic topology which we define next. A coboundary has the key property that for topologically trivial domains, its integral depends only on data defined on the boundary of the domain of integration. It is the generalisation, to an arbitrary mesh, of a telescoping sum.

Definition 1.4.2 *Let X be a simplicial (triangulated) space. Denote by $\bar{\mathcal{C}}_n(\mathcal{R})$ the vector space formed by all formal, finite sums of the n-simplexes of X with coefficients in $\mathcal{R}$. There is a boundary map*

$$\partial : \bar{\mathcal{C}}_n \longrightarrow \bar{\mathcal{C}}_{n-1}$$

obtained by mapping each simplex to the sum of its boundary edges, signed according to whether the orientation of the edge is that induced by the orientation of the simplex or its opposite, and extended linearly.

Example 1.4.3 *In Figure 1.1 we show an oriented simplex τ together with its oriented edges e_i. The boundary $\partial\tau = e_1 - e_2 + e_3$, where the signs are determined by whether the orientation on τ induces the given orientation on the edge, or not.*

See Frankel (1997), Chapter 13 (in particular 13.2b), for a readable account of oriented chains and the boundary map.

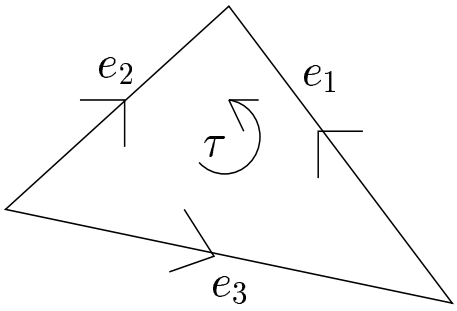


Fig. 1.1. $\partial(\tau) = e_1 - e_2 + e_3$

Definition 1.4.4 *For the simplicial space X, an n-cochain with coefficients in $\mathcal{R}$ is a map $\phi : \bar{\mathcal{C}}_n \longrightarrow \mathcal{R}$. The set of simplicial n-cochains is denoted $\bar{\mathcal{C}}^n$. The coboundary map*

$$\delta : \bar{\mathcal{C}}^n \longrightarrow \bar{\mathcal{C}}^{n+1}$$

is defined by

$$(\delta\phi)(\sigma) = \phi(\partial\sigma).$$

For the simplex in Example 1.4.3, if $\phi(e_i) = c_i$, then $(\delta\phi)(\tau) = c_1 - c_2 + c_3$.

If a cochain ψ is of the form $\delta\phi$ for some cochain ϕ, we say simply that ψ *is* a coboundary. If the simplicial space is in fact a regular triangulation, it will be possible to write coboundaries in the form of a total difference.

For the variational calculations that we consider, the coefficients $\mathcal{R}$ are vertical forms in the $\mathrm{d}\alpha^\tau$ which themselves have coefficient functions of the moments, the mesh variables, and so forth.

We can finally define the Finite Element Euler–Lagrange operator to be

$$E = \pi \circ \widehat{\mathrm{d}} \circ \int$$

where $\int$ is the integration over each p-dimensional simplex that is used to obtain the projected Lagrangian, and π is the projection map to equivalence classes, where two forms are equivalent if they differ by a p-dimensional coboundary. (Recall p is the dimension of the base space.)

The variational complex for the Finite Element Method (Mansfield and Quispel (2005)), is then

$$\xrightarrow{\mathrm{d}} \quad \widetilde{\mathcal{F}}^{p-1} \quad \xrightarrow{\mathrm{d}} \quad \widetilde{\mathcal{F}}^{p} \quad \xrightarrow{E} \quad \mathcal{F}^1_* \quad \xrightarrow{\mathrm{d}_*} \quad \mathcal{F}^2_* \quad \xrightarrow{\mathrm{d}_*}$$

where:

- $\widetilde{\mathcal{F}}_*$ is the algebra generated by the $\mathcal{F}_i$ with unevaluated degrees of freedom
- $\mathrm{d}_* = \pi \circ \widehat{\mathrm{d}}$ is the vertical exterior derivative, that is, with respect to the degrees of freedom, modulo coboundaries,
- $\mathcal{F}^*_*$ is the algebra of vertical forms modulo p-dimensional coboundaries.

Looking now at the analogue of (1.6) and (1.12) for the Finite Element variational complex, we can tentatively write

$$\longrightarrow \widetilde{\mathcal{F}}^p \underset{v_Q \lrcorner}{\overset{E}{\underset{\longleftarrow}{\longrightarrow}}} \widetilde{\mathcal{F}}^1_* \longrightarrow \tag{1.17}$$

Taking $L_\tau \in \widetilde{\mathcal{F}}^p$, if the natural symmetry condition holds, that is $v_Q \lrcorner \hat{\mathrm{d}}(L_\tau) = 0$ (or, more generally, is a coboundary), we will have the **Finite Element Noether's Theorem**,

$$0 = \sum_\tau Q_\tau \cdot E(L_\tau) + \delta(\eta(L, Q)). \tag{1.18}$$

There are two problems. One is to find the general formula for both E and $\eta(L, Q)$ for an arbitrary mesh. If the mesh is regular, then E and $\eta(L, Q)$ will be easily derivable from arguments analogous to those for the difference case. There are increasingly many computational arguments in favour of considering *cubical* simplicial spaces; see Kaczynski, Mischaikow and Mrozek (2003), for an exposition. In this case, coboundaries are essentially total differences.

The second problem is to define v_Q, which requires determining the infinitesimal action that is induced on the moments and other degrees of freedom. In the next section, we address this second problem.

1.4.3 Group actions on moments

For degrees of freedom that are values of a function at a particular point, the induced action is the same as for the function itself, and the discussion in Section 1.3.2 applies. For degrees of freedom that are moments defined by integrals, we can use results for the variational symmetry group action on Lagrangians derived earlier. Thus, given a group action on the independent and dependent variables, we take as

a definition of the induced group action on the moment with weight function ψ,

$$\epsilon \cdot \int_\tau f(x,u,\cdots)\psi(x)\,\mathrm{d}x = \int_\tau f(\epsilon\cdot x,\epsilon\cdot u,\cdots)\psi(\epsilon\cdot x)\frac{\mathrm{D}\epsilon\cdot x}{\mathrm{D}x}\,\mathrm{d}x. \quad (1.19)$$

Example 1.4.5 *Suppose the group action is translation in x, so that $\epsilon\cdot x = x+\epsilon$, while the dependent variables are invariants, $\epsilon\cdot u = u$. Then the induced action on the moments*

$$\alpha^j = \int_\tau x^j u\,\mathrm{d}x \quad (1.20)$$

is

$$\begin{aligned}\epsilon\cdot\alpha^0_\tau &= \alpha^0_\tau,\\ \epsilon\cdot\alpha^1_\tau &= \alpha^1_\tau + \epsilon\alpha^0_\tau,\\ \epsilon\cdot\alpha^2_\tau &= \alpha^2_\tau + 2\epsilon\alpha^1_\tau + \epsilon^2\alpha^0_\tau,\end{aligned}$$

and so forth. Thus,

$$Q^j_\tau = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0}(\epsilon\cdot\alpha^j_\tau) = j\alpha^{j-1}_\tau,$$

whence

$$\mathbf{v}_Q = \sum_{\tau,j} j\alpha^{j-1}_\tau\frac{\partial}{\partial\alpha^j_\tau}.$$

If the mesh variables x_n are also regarded as dependent (i.e. movable) then one would add

$$\sum_n \frac{\partial}{\partial x_n}$$

to this vector. The zeroth and first-order moment invariants are generated by

$$\alpha^0_\tau, \quad \alpha^0_\tau\alpha^1_\sigma - \alpha^1_\tau\alpha^0_\sigma$$

while the second-order invariants are generated by

$$\alpha^2_\tau(\alpha^0_\sigma)^2 - 2\alpha^1_\sigma\alpha^1_\tau\alpha^0_\tau + (\alpha^1_\tau)^2\alpha^0_\sigma,$$

where τ and σ are not necessarily distinct simplexes. The method of moving frames shows that any moment invariant (to order two) is a function of these. Allowing movable mesh variables, we have that $x_n - x_m$ is an invariant, as is $\alpha^0_\tau x_n - \alpha^1_\tau$.

1.4.4 Building in a conservation law

The algebra underpinning Noether's theorem shows that designing a conservation law into a numerically stable scheme requires a number of conditions to be met on the choice of moments and how the Lagrangian is approximated.

- The approximation of forms is required to fit into a commutative diagram, (1.16) not only for stability but for the variational complex to be applicable.
- The induced group action probably should involve essentially a finite number of moments, so it may be necessary to use symmetry-adapted moments. For example, if the group action is the projective action, (1.1), then the index j in the moments (1.20) needs to be in the range $-N, \cdots, -3$.
- The projected Lagrangian form needs to be invariant under the induced action and have the correct continuum limit.

1.5 Conclusions

Instead of the approximate conservation of an exact law, the algebraic arguments offered here yield exact conservation of an approximate law! Clearly much remains to be done to bring these ideas into the practical arena, in particular the analytic problem of achieving everything listed in Section 1.4.4 for some interesting applications. Another problem is, how does the order of approximation of the conservation law compare to the order of approximation of the scheme? Whether such schemes prove efficient and useful is for the future to decide. Nevertheless, schemes with guaranteed conservation laws appear to be possible.

Acknowledgments

It is a pleasure to acknowledge the support and encouragement of Arieh Iserles and Peter Olver, discussions with Peter Hydon and Reinout Quispel, financial assistance from the Leverhulme Trust, UK and the hospitality of the Institute of Advanced Studies, LaTrobe University, Australia.

Bibliography

Arnold D., (2002), *Differential complexes and numerical stability*, Proceedings of the International Congress of Math-

ematicians, Beijing 2002, Volume I: Plenary Lectures. `http://www.ima.umn.edu/~arnold/papers/icm2002.pdf`

Bila N., Mansfield E.L. and Clarkson P.A., (2005) Symmetry Group Analysis of the Shallow Water and Semi-Geostrophic Systems *Q. J. Mech. Appl. Math.*, to appear.

Brezzi F., (1974) On the existence, uniqueness and approximation of saddle point problems arising from Lagrange multipliers, *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge Anal. Numér.*, **8** 129–151.

Chard J.A. and Shapiro V., (2000) A Multivector Data Structure for Differential Forms and equations, *Math. Comp. Sim.* **54** 33–64

Cheb-Terrab E.S., Duarte L.G.S. and da Mota L.A., (1998) Computer Algebra Solving of Second Order ODEs Using Symmetry Methods, *Computer Physics Communications* **108** 90–114.

Dorodnitsyn V., (2001) Noether-type theorems for difference equations, *App. Num. Math.* **39** 307–321

Fels M. and Olver P.J., (1998) Moving Coframes I *Acta Appl. Math.* **51** 161–213

Fels M. and Olver P.J., (1999) Moving Coframes II *Acta Appl. Math.* **55** 127–208

Frankel T., (1997) The Geometry of Physics, Cambridge University Press, Cambridge.

Hereman W., (1997) Review of symbolic software for Lie symmetry analysis *Math. Comp. Model.* **25** 115–132

Hereman W., Colagrosso M., Sayers R., Ringler A., Deconinck B., Nivala M. and Hickman M.S, (2005) *Continuous and Discrete Homotopy Operators with Applications in Integrability Testing.* In: Differential Equations with Symbolic Computation, Eds.: D. Wang and Z. Zheng, World Scientific Publishing Company, Singapore.

Hereman W., Sanders J.A., Sayers J. and Wang J.P., (2005) *Symbolic computation of polynomial conserved densities, generalized symmetries, and recursion operators for nonlinear differential-difference equations.* In: Group Theory and Numerical Analysis, Eds.: P. Winternitz, D. Gomez-Ullate, A. Iserles, D. Levi, P.J. Olver, R. Quispel and P. Tempesta, CRM Proceedings and Lecture Notes **39**, American Mathematical Society, Rhode Island.

Hiptmair R., (2002) Finite elements in computational electromagnetism, *Acta Numerica* **11** 237–340

Hubert E., (2000) Factorization free decomposition algorithms in differential algebra *J. Symb. Comp.* **29** 641–662

Hubert E., (2003) *Notes on triangular sets and triangulation-decomposition algorithms II: Differential Systems*, in: Symbolic and Numerical Scientific Computations [Editors U. Langer and F. Winkler] *Lect. Notes Comp. Sci.*, **2630**, Springer-Verlag Heidelberg, 40–87.

Hydon P.E., (2000) Symmetries and first integrals of ordinary difference equations *Phil. Trans. R. Soc. Lond. A* **456** 2835–2855.

Hydon P.E. and Mansfield E.L., (2004) A variational complex for difference equations *Foundations of Computational Mathematics* **4** 187–217.

Kaczynski T., Mischaikow K. and Mrozek M., (2003) Computing Homology *Homology, Homotopy and Applications* **5** 233–256.

Kupershmidt B, (1985) *Discrete Lax equations and differential-difference calculus. Astérisque* **123**, SMF, Paris.

Lee T.D., (1987) Difference Equations and Conservation Laws, J. Stat. Phys., **46** 843–860.

Levi D., Tremblay S. and Winternitz P., (2005) Lie symmetries of multidimensional difference equations, *J. Phys. A: Math. Gen.* **34** 9507–9524.

Mansfield E.L. and Clarkson P.A., (1997) Applications of the differential algebra package `diffgrob2` to classical symmetries of differential equations *J. of Symb. Comp.* **23** 517–533.

Mansfield E.L. and Hydon P.E., (2001) Towards approximations which preserve integrals, in: ISSAC 2001, Editor B. Mourrain, ACM Publishing, New York, 217–222

Mansfield E.L. and Quispel G.R.W., (2005) *Towards a variational complex for the Finite Element Method* In: Group Theory and Numerical Analysis, Eds.: P. Winternitz, D. Gomez-Ullate, A. Iserles, D. Levi, P.J. Olver, R. Quispel and P. Tempesta, CRM Proceedings and Lecture Notes **39**, American Mathematical Society, Rhode Island.

Mattiussi C., (1997) An analysis of finite volume, finite element, and finite difference methods using some concepts from algebraic topology, *J. Comp. Phys***133** 289–309.

Olver P.J., (1993) Applications of Lie Groups to Differential Equations *Graduate Texts Math.*, **107**, Second Edition, Springer Verlag, New York.

Olver, P.J., (2001) Geometric foundations of numerical algorithms and symmetry, *Appl. Alg. Engin. Comp. Commun.* **11** 417–436.

Quispel G.R.W., Capel H.W. and Sahadevan R., (1992) Continuous symmetries of differential-difference equations – the Kac-van Moerbeke equation and Painlevé reduction *Phys. Lett.* **170A** 379–383.

Reid G.J., (1991) Algorithms for reducing a system of PDEs to standard form, determining the dimension of its solution space and calculating its Taylor series solution *Europ. J. Appl. Math.* **2** 293–318.

Reid G.J., Wittkopf A.D. and Boulton A., (1996) Reduction of systems of nonlinear partial differential equations to simplified involutive forms *Europ. J. Appl. Math.* **7** 604–635

Schwalm W., Moritz B., Giona M., Schwalm M., (1999) Vector difference calculus for physical lattice models, *Phys Rev E* **59** 1217-1233.

Tonti E., (1975) *On the formal structure of physical theories*, Istituto di Matematica del Politecnico di Milano, Milan.
`http://www.dic.univ.trieste.it/perspage/tonti/papers.htm`

Wolf T., (2000) The Symbolic Integration of Exact PDEs *J. Symb. Comp.* **30** 619–629

Wolf T., (2002) A comparison of four approaches to the calculation of conservation laws *Europ. J. Appl. Math.* **13** 129–152

Wolf T., Brand A. and Mohammadzadeh M., (1999) *Computer algebra algorithms and routines for the computation of conservation laws and fixing of gauge in differential expressions J. Symb. Comp.* **27** 221–238.