Nonlinear Perron-Frobenius theory and dynamics of cone maps

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Summary. In this paper several recent results concerning the dynamics of order preserving (sub)homogeneous maps on polyhedral cones are reviewed. These results were obtained by the author in collaboration with Marianne Akian, Stéphane Gaubert, Roger Nussbaum, Michael Scheutzow and Colin Sparrow in [2], [13] and [15], and are new nonlinear extensions of the Perron-Frobenius theory.

1 The classical Perron-Frobenius theorem

To start this review it is convenient to first recall the classical Perron-Frobenius theorem, which can be stated as follows (cf. [5]).

Theorem 1 (Perron-Frobenius). *If A is a nonnegative irreducible $n \times n$ matrix, then the following assertions hold:*

(i) The spectral radius, $\rho(A)$, is a simple eigenvalue of A and the corresponding eigenvector is unique (up to scaling) and positive.
(ii) If, in addition, A has exactly q distinct eigenvalues λ such that $|\lambda| = \rho(A)$, then these eigenvalues are; $\rho(A)e^{2\pi k/q}$ for $k = 0, \ldots, q-1$.

The Perron-Frobenius theorem has the following consequence (cf. [19]) for the dynamical behavior of linear maps on the *standard positive cone*,

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \colon x_i \geq 0 \text{ for all } 1 \leq i \leq n\}.$$

Recall that a point $x \in X$ is a *periodic point* of a map $f \colon X \to X$ if there exists an integer $p \geq 1$ such that $f^p(x) = x$ and the minimal such p is called the *period* of x under f.

Corollary 1. *If $A \colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is a linear map, then there exists an integer $p \geq 1$ such that for each $x \in \mathbb{R}^n_+$ with $(\|A^k x\|)_k$ bounded, $\lim_{k\to\infty} A^{kp}x = \xi_x$, where ξ_x is a periodic point of A whose period divides p. Moreover, p is the order of a permutation on n letters.*

In other words, each bounded orbit of A in the standard positive cone converges to a periodic orbit of A whose period divides the order of a permutation on n letters.

It is well known that the first assertion in the Perron-Frobenius theorem can be generalized to linear maps that leave a cone in $\mathbb{R}^n$ invariant (see [5] or [12]). However, to generalize the second assertion one needs the cone to be polyhedral, i.e., it has finitely many extremal rays (see [12]). Indeed, if we consider the Lorentz cone, $L_3 = \{x \in \mathbb{R}^3 \colon x_1^2 + x_2^2 \leq x_3^2 \text{ with } x_3 \geq 0\}$, and

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then A leaves L_3 invariant and the spectrum of A is $\{1, e^{\pm\theta i}\}$. If θ is an irrational multiple of 2π, then $e^{\theta i}$ is not a root of unity and hence the second assertion of the Perron-Frobenius theorem does not hold. Moreover, in that case, all orbits of A in L_3 are bounded, but most of them do not converge to periodic orbits.

In [2], [13] and [15] we were particularly interested in finding nonlinear generalizations of the Perron-Frobenius theorem for order preserving (sub)homogeneous maps on polyhedral cones. Such maps arise in a variety of applications, including optimal control and game theory [1, 20], computer science [4], mathematical biology [12, 21] and in the analysis of discrete event systems [3, 8]. Their eigenvectors and dynamical behavior have been investigated in numerous papers, see for instance [1, 7, 9, 10, 11, 16, 17, 18, 22, 24] and the references therein. In connection with Perron-Frobenius theory the following questions are particularly interesting. When does an order preserving (sub)homogeneous map have an eigenvector in the interior of the cone and when it is unique? Does every bounded orbit of an order preserving (sub)homogeneous map on a polyhedral cone converge to a periodic orbit and, if so, what are the possible periods? Our results provide a detailed answer for the second question. Before stating the results precisely we first recall several basic definitions.

2 Basic definitions and examples

A *cone* $K \subseteq \mathbb{R}^n$ is a convex subset of $\mathbb{R}^n$ such that $\lambda K \subseteq K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{0\}$. It is said to be a *closed* cone if it is a closed subset of $\mathbb{R}^n$. The interior and boundary of a cone K are denoted by K° and ∂K, respectively. A closed cone $K \subseteq \mathbb{R}^n$ is called *polyhedral* if there exist finitely many linear functionals, $\phi_1, \ldots, \phi_m$ such that $K = \{x \in \mathbb{R}^n \colon \phi_i(x) \geq 0 \text{ for all } 1 \leq i \leq m\}$. A *face* of a polyhedral cone K is any set of the form $F = K \cap \{x \in \mathbb{R}^n \colon \phi(x) = 0\}$, where ϕ is a linear functional on $\mathbb{R}^n$ such that $K \subseteq \{x \in \mathbb{R}^n \colon \phi(x) \geq 0\}$. A face is called a *facet* if $\dim(F)$, the dimension of the linear span of F, is equal to $\dim(K)-1$. For instance, the standard positive cone, $\mathbb{R}^n_+$, has 2^n faces and n facets.

A cone $K \subset \mathbb{R}^n$ induces a partial ordering $\leq$ on $\mathbb{R}^n$ by, $x \leq y$ if $y-x \in K$. We call a map $f\colon K \to K$ *order preserving* if $f(x) \leq f(y)$ whenever $x \leq y$. A map $f\colon K \to K$ is said to be *subhomogeneous* if $\lambda f(x) \leq f(\lambda x)$ for all $x \in K$ and $0 \leq \lambda \leq 1$. It is called *homogeneous* if $\lambda f(x) = f(\lambda x)$ for all $x \in K$ and $\lambda \geq 0$.

There are many examples of order preserving (sub)homogeneous maps. Consider, for instance, an *affine map* $f\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ given by, $f(x) = Ax+b$ for $x \in \mathbb{R}^n_+$, where A is a nonnegative matrix and $b \in \mathbb{R}^n_+$. It is easy to verify that such affine maps are order preserving and subhomogenous. Other important examples are so called min-max maps. Before defining these maps it is useful to introduce the following notation: $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$ for $a,b \in \mathbb{R}$. A *min-max map* is map $f\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ of the form

$$f_i(x) = \bigvee_{1\leq j\leq m} \bigwedge_{k\in I_j} a^i_{jk} x_k \quad \text{for } 1 \leq i \leq n \text{ and } x \in \mathbb{R}^n_+,$$

where $I_j \subseteq \{1,\ldots,n\}$ and each $a^i_{jk} > 0$. (Here the integer m and the sets I_j may be different for different i.) Other examples are maps $f\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ given by $f(x) = \sup_{P\in\mathcal{P}} Px$ for $x \in \mathbb{R}^n_+$, where $\mathcal{P}$ is a collection of non-negative matrices and the supremum is taken coordinate-wise. Another rich source of examples is provided by order preserving additively homogeneous maps, where the ordering is induced by $\mathbb{R}^n_+$. Recall that a map $g\colon \mathbb{R}^n \to \mathbb{R}^n$ is *additively homogeneous* if $g(x+\lambda\mathbf{1}) = g(x)+\lambda\mathbf{1}$ for all $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. (Here $\mathbf{1}$ denotes the vector with each coordinate unity.) These maps become, after a transformation, order preserving and homogenous on $(\mathbb{R}^n_+)^\circ$. Indeed, if $g\colon \mathbb{R}^n \to \mathbb{R}^n$ is an order preserving additively homogeneous map, then we can consider the map $f\colon (\mathbb{R}^n_+)^\circ \to (\mathbb{R}^n_+)^\circ$ given by

$$f(x) = (\exp \circ\, g \circ \log)(x) \quad \text{for } x \in (\mathbb{R}^n_+)^\circ,$$

where $\log(z) = (\log z_1,\ldots,\log z_n)$ and $\exp(z) = (e^{z_1},\ldots,e^{z_n})$ for all z. It is easy to verify that f preserves the ordering induced by $\mathbb{R}^n_+$ and f is homogeneous. Maps that are order preserving and additively homogeneous arise frequently in the analysis of discrete event systems, see [3] and [8], and include max-plus maps and min-max-plus maps.

3 Non-expansiveness and Thompson's metric

It is known that order preserving (sub)homogeneous maps are non-expansive with respect to Thompson's metric. Before defining this metric we first introduce the notion of a part of the cone. The partial ordering $\leq$ yields an equivalence relation $\sim$ on K by, $x \sim y$ if there exist $0 < \alpha \leq \beta$ such that

$\alpha x \leq y \leq \beta x$. The equivalence classes are called *parts* of the cone and the set of all parts of K is denoted by $\mathcal{P}(K)$. It can be shown (see [2]) that if K is a polyhedral cone with N facets, then $|\mathcal{P}(K)| \leq 2^N$. In particular, $\mathbb{R}^n_+$ has exactly 2^n parts, which are given by

$$P_I = \{x \in \mathbb{R}^n_+ \colon x_i > 0 \text{ for all } i \in I \text{ and } x_i = 0 \text{ otherwise}\} \quad \text{for } I \subseteq \{1, \ldots, n\}.$$

Now for $x, y \in K$ we define $M(x/y) = \inf\{\beta > 0 \colon x \leq \beta y\}$ and we put $M(x/y) = \infty$ if the set is empty. By using the function $M(x/y)$ one can define *Thompson's metric*, $d_T \colon K \times K \to [0, \infty]$, on a closed cone K as follows;

$$d_T(x, y) = \log(M(x/y) \vee M(y/x))$$

for all $(x, y) \in K \times K$ with $(x, y) \neq (0, 0)$ and $d_T(0, 0) = 0$. It is not hard to show that d_T is a genuine metric on each part of the closed cone and $d_T(x, y) < \infty$ if and only if $x \sim y$. Moreover, if P is a part of a closed cone in $\mathbb{R}^n$, then (P, d_T) is a complete metric space and the topology coincides with the norm topology (see [17] or [23]).

Recall that $f \colon K \to K$ is called *non-expansive under* d_T if

$$d_T(f(x), f(y)) \leq d_T(x, y) \quad \text{for all } x, y \in K.$$

The following lemma shows that order preserving subhomogeneous maps are non-expansive under d_T (cf. [2, Lemma 3.3]).

Lemma 1. *Let K be a closed cone in $\mathbb{R}^n$. If $f \colon K \to K$ is order preserving, then f is non-expansive under d_T if and only if f is subhomogeneous.*

Proof. Suppose that f is subhomogeneous. Let $x, y \in K$ and assume that $\lambda = \max\{M(y/x), M(x/y)\}$. Then $y \leq \lambda x$ and $x \leq \lambda y$. This implies that $x \leq \lambda y \leq \lambda^2 x$ and hence $\lambda \geq 1$. As f is order preserving and subhomogeneous, we deduce that $\lambda^{-1} f(y) \leq f(\lambda^{-1} y) \leq f(x)$ and $\lambda^{-1} f(x) \leq f(\lambda^{-1} x) \leq f(y)$. Therefore $\max\{M(f(y)/f(x)), M(f(x)/f(y))\} \leq \lambda$ and hence $d_T(f(x), f(y)) \leq \log \lambda = d_T(x, y)$.

Now suppose that f is nonexpansive with respect to d_T on K. Let $x \in K$ and put $y = \lambda^{-1} x$, where $\lambda \geq 1$. Clearly $d_T(x, y) = \log \lambda$ if $x \neq 0$ and $d_T(x, y) \leq \log \lambda$ if $x = 0$. As f is nonexpansive with respect to d_T, we deduce that $\log M(f(x)/f(y)) \leq d_T(f(x), f(y)) \leq d_T(x, y) \leq \log \lambda$, so that $f(x) \leq \lambda f(y)$. This implies that $\lambda^{-1} f(x) \leq f(y) = f(\lambda^{-1} x)$ and hence f is subhomogeneous.

There exists a close relation between Thompson's metric and the sup-norm on $\mathbb{R}^n$ given by $\|z\|_\infty = \max_i |z_i|$ for $z \in \mathbb{R}^n$. Indeed, if we let $K = \mathbb{R}^n_+$ and $x, y \in (\mathbb{R}^n_+)^\circ$, then

$$M(x/y) = \inf\{\beta > 0 \colon x \leq \beta y\} = \max\{x_i / y_i \colon 1 \leq i \leq n\},$$

so that $\log M(x/y) = \max_i (\log x_i - \log y_i)$. As $\|z\|_\infty = (\max_i z_i) \vee (\max_i -z_i)$ for all $z \in \mathbb{R}^n$, we find that $\log \colon (\mathbb{R}^n_+)^\circ \to \mathbb{R}^n$ is an isometry from $((\mathbb{R}^n_+)^\circ, d_T)$

onto $(\mathbb{R}^n, \|\cdot\|_\infty)$. Thus, if $f\colon (\mathbb{R}^n_+)^\circ \to (\mathbb{R}^n_+)^\circ$ is an order preserving subhomogenous map, where the ordering is induced by $\mathbb{R}^n_+$, then $g\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $g = \log \circ f \circ \exp$, is a sup-norm non-expansive map that has the same dynamical properties as f. In case K is a polyhedral cone with nonempty interior and N facets, we can use the facet defining functionals to construct an isometric embedding of (K°, d_T) into $(\mathbb{R}^N, \|\cdot\|_\infty)$ (see [2, Section 4]).

The dynamics of sup-norm non-expansive maps is fairly well understood. In fact, there exists the following theorem.

Theorem 2. *If $f\colon X \to X$, where $X \subseteq \mathbb{R}^n$ is closed, is a sup-norm non-expansive map and there exists $z \in X$ such that $(\|f^k(z)\|_\infty)_k$ is bounded, then for each $x \in X$ there exist an integer $p \geq 1$ and a periodic point $\xi_x \in X$ of f with period p such that $(f^{kp}(x))_k$ converges to ξ_x and p does not exceed $\max_k 2^k\binom{n}{k}$.*

The first statement in Theorem 2 was proved by Weller [24]. The estimate for p was obtained by the author and Michael Scheutzow in [13]. It has been conjectured by Nussbaum [18] that the optimal upper bound for p in Theorem 2 is 2^n. At present this conjecture is know to be true for $1 \leq n \leq 3$. The estimate in Theorem 2 is smaller than $C3^n/\sqrt{n}$ and is currently the strongest estimate. Evidence supporting Nussbaum's conjecture is given in [14].

As order preserving subhomogenous maps on the interior of a polyhedral cone have the same dynamical properties as sup-norm non-expansive maps, Theorem 2 has the following consequence. If $f\colon K \to K$ is an order preserving subhomogeneous map on a polyhedral cone, with nonempty interior and N facets, and there exists $z \in K^\circ$ whose orbit has a compact closure in K°, then the orbit of each $x \in K^\circ$ converges to a periodic orbit of f in K° and its period is at most $\max_k 2^k\binom{N}{k}$. This consequence only concerns a special situation. It does, for instance, not provide any information about orbits in K° that are bounded but have limit points at ∂K or orbits that are contained in ∂K. In the next section a more complete picture of the possible dynamics is given.

4 Nonlinear Perron-Frobenius theorems

We start by discussing the periodic orbits in the interior of the cone in more detail. As order preserving subhomogenous maps on polyhedral cones are more constrained than sup-norm non-expansive maps, one may expect to obtain a stronger estimate for the periods of periodic orbits in K°. In fact, there exists the following result from [13], which confirmed a conjecture of Gunawardena and Sparrow [8].

Theorem 3. *If $f\colon (\mathbb{R}^n_+)^\circ \to (\mathbb{R}^n_+)^\circ$ is an order preserving homogeneous map, where the ordering is induced by $\mathbb{R}^n_+$, then the period of each periodic point of f does not exceed $\binom{n}{\lfloor n/2 \rfloor}$.*

It is known that the upper bound in Theorem 3 is sharp (see [8]). The main idea of the proof of Theorem 3 can be summarized as follows. Let $\mathcal{O} \subset (\mathbb{R}^n_+)^\circ$ be a periodic orbit of an order preserving homogeneous map f. An ordered pair $(x, y) \in \mathcal{O} \times \mathcal{O}$, with $x \neq y$, is called an *extreme pair in* $\mathcal{O}$ if there exists no $z \in \mathcal{O}$ with $x \neq z \neq y$ such that either $\log M(z/y) = \log M(z/x) + \log M(x/y)$ or $\log M(x/z) = \log M(x/y) + \log M(y/z)$. By using the extreme pairs, a coding function $c\colon \mathcal{O} \to 2^{\{1,\ldots,n\}}$ can be defined as follows: $i \in c(x)$ if and only if there exists $y \in \mathcal{O}$ such that (x, y) an extreme pair in $\mathcal{O}$ and $M(x/y) = x_i/y_i$. It can be shown that c is injective and that $c(\mathcal{O})$ is an anti-chain in the partially ordered set $(2^{\{1,\ldots,n\}}, \subseteq)$, i.e., there exist no $A, B \in c(\mathcal{O})$ such that $A \subseteq B$ and $A \neq B$. As such anti-chains have at most $\binom{n}{\lfloor n/2 \rfloor}$ elements by Sperner's Theorem [6, §3], the conclusion of Theorem 3 follows.

Theorem 4 has a generalization to order preserving subhomogeneous maps on polyhedral cones (cf. [2, Theorem 4.2]).

Theorem 4. *Let K be a polyhedral cone in $\mathbb{R}^n$ with nonempty interior and N facets. If $f\colon K^\circ \to K^\circ$ is an order preserving and subhomogeneous map, where the ordering is induced by K, then the period of each periodic point of f does not exceed $\binom{N}{\lfloor N/2 \rfloor}$.*

Let us now consider bounded orbits of order preserving subhomogeneous maps in the whole polyhedral cone. Since points in different parts of the cone are at infinite distance from each other in Thompson's metric, we can not rely on non-expansiveness only. Nevertheless one can still show that every bounded orbit converges to a periodic orbit (see [2]). In fact, the following theorem is true.

Theorem 5. *Let K be a polyhedral cone with N facets in $\mathbb{R}^n$. If $f\colon K \to K$ is a continuous order preserving subhomogeneous map and the orbit of $x \in K$ is bounded, then there exists a periodic point ξ_x of f, with period p, such that $(f^{kp}(x))_k$ converges to ξ_x. Moreover, there exist an integer $1 \leq m \leq N$ such that $p = q_1 q_2$ for some integers $1 \leq q_1 \leq \binom{N}{m}$ and $1 \leq q_2 \leq \binom{m}{\lfloor m/2 \rfloor}$.*

It follows from Theorem 5 that the period of each periodic point of an order preserving subhomogeneous map on a polyhedral cone with N facets is bounded by

$$\beta_N = \max_{1 \leq m \leq N} \binom{N}{m} \binom{m}{\lfloor m/2 \rfloor} = \frac{N!}{\lfloor \frac{N}{3} \rfloor! \lfloor \frac{N+1}{3} \rfloor! \lfloor \frac{N+2}{3} \rfloor!} \sim \frac{3^{N+1}\sqrt{3}}{2\pi N}.$$

The intuition behind this bound can be explained as follows. To begin we remark that, as f is non-expansive under d_T, $x \sim y$ implies that $f(x) \sim f(y)$. Therefore f has to map parts into parts. This is a severe constraint on the periodic orbits in the boundary of the cone. For instance, in Figure 1 below two period 6 orbits in the boundary of $\mathbb{R}^3_+$ are depicted, but only the right one may occur as a periodic orbit of an order preserving subhomogeneous map on $\mathbb{R}^3_+$. This observation allows us to define a quotient map $F\colon \mathcal{P}(K) \to \mathcal{P}(K)$

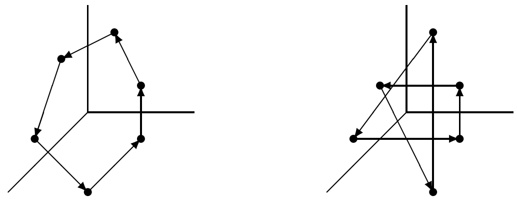

Fig. 1. Period 6 orbits in $\mathbb{R}^3_+$

by $F([x]) = [f(x)]$, where $[x]$ denotes the equivalence class (the part of) x in K. On $\mathcal{P}(K)$ we also have a partial ordering $\preceq$ given by, $P \preceq Q$ if there exist $x \in P$ and $y \in Q$ such that $x \leq \beta y$ for some $\beta > 0$. It is not hard to show that F preserves the ordering $\preceq$ on $\mathcal{P}(K)$, since f is order preserving and subhomogeneous. Now, if $x \in K$ is a periodic point of f with period p, then $[x]$ is a periodic point of F with period say q_1. Moreover, each part in the orbit of $[x]$ under F contains the same number of points of the orbit of x, say q_2. Thus, we can write $p = q_1 q_2$, where q_1 is the number of parts visited by the orbit of x and q_2 is the number of points of the orbit in each of these parts. Only a limited number of parts can be visited by the orbit of x, since the orbit of $[x]$ under F is an anti-chain in the partially ordered set $(\mathcal{P}(K), \preceq)$. On the other hand, the smallest dimension among the parts visited by the periodic orbit limits the number of points of the orbit contained in each part. Using these observations one can prove the existence of an integer $1 \leq m \leq N$ such that $1 \leq q_1 \leq \binom{N}{m}$ and $1 \leq q_2 \leq \binom{m}{\lfloor m/2 \rfloor}$.

Knowing Theorem 5 it is natural to define for a polyhedral cone K the set $\Gamma(K)$ as the set of all possible periods of periodic points of continuous, order preserving, subhomogeneous map $f\colon K \to K$. It follows from Theorem 5 that if K has N facets, then $\Gamma(K)$ is contained in the set $B(N)$ consisting of those $p \geq 1$ for which there exist integers q_1 and q_2 such that $p = q_1 q_2$, $1 \leq q_1 \leq \binom{N}{m}$, and $1 \leq q_2 \leq \binom{m}{\lfloor m/2 \rfloor}$ for some $1 \leq m \leq N$. In particular, we find that $\Gamma(\mathbb{R}^n_+) \subseteq B(n)$. It is natural to ask if we can completely determine the set $\Gamma(\mathbb{R}^n_+)$ for each $n \geq 1$ and, in particular, if $\Gamma(\mathbb{R}^n_+) = B(n)$. The next result from [15] says that the last equality does indeed hold.

Theorem 6. *For each* $n \geq 1$, $\Gamma(\mathbb{R}^n_+) = B(n)$.

It turns out that for each $p \in B(n)$ a min-max map $f\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ can be constructed which has a periodic point with period p. For instance, the min-max map $f\colon \mathbb{R}^3_+ \to \mathbb{R}^3_+$ given by

$$f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} (3x_1 \wedge x_2) \vee (3x_2 \wedge x_3) \\ (3x_1 \wedge x_3) \vee (x_2 \wedge 3x_3) \\ (x_1 \wedge 3x_2) \vee (x_1 \wedge 3x_3) \end{pmatrix}$$

has the point $(1,2,0)$ as a period 6 point. We conclude the review with a small table listing the elements of $B(n)$ for $1 \leq n \leq 5$.

$B(n)$
1
1,2
1,2,3,4,6
1,2,3,4,5,6,8,9,12
1,2,3,4,5,6,7,8,9,10,12,14,15,16,18,20,21,24,25,27,30

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