Integration

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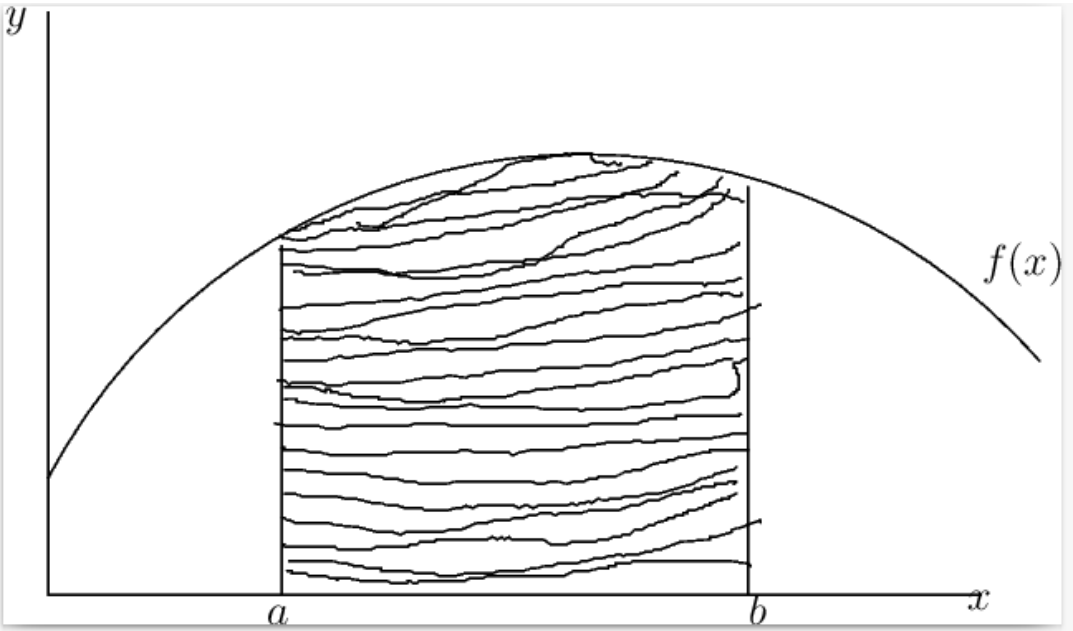
Introduction

Let $f(x)$ be a function whose graph is given below. How can one find the area of the shaded region?



Answer: Integration.

Hence, **integration** can be described as the process of finding the area under a curve.

$$\text{Area} = \int_a^b f(x)dx.$$

- a is called the lower limit of the integration.
- b is called the upper limit of the integration.
- $f(x)$ is called the integrand.
- dx indicates that the integration is done wrt the variable x.

Another scenario!

Let $y = x^2 + c$, where c is a constant. We know that $\dfrac{dy}{dx} = 2x$.

Que: How can one recover the function y from $\dfrac{dy}{dx}$?

Ans: Integration.

Hence, integration can also be described as the process of finding the **antiderivative** of a function.

Note, $\int f(x)dx$ is the antiderivative of the function $f(x)$.

What is the difference between $\int_a^b f(x)dx$ and $\int f(x)dx$?

$\int_a^b f(x)dx$

- Has limits.
- It's constant when a and b are constant.
- Called integration with limits (definite integral).

$\int f(x)dx$

- Has no limits.
- It's a family of functions.
- Called integration without limits (indefinite integral).

Let $f(x) = ax^n$, then

$$\int f(x)dx = \int ax^n dx = a\int x^n dx = \frac{a}{n+1}x^{n+1} + c,$$

where a, c are constants.

Let $f(x) = cg(x) \pm h(x)$, then

$$\int f(x)dx = c\int g(x)dx \pm \int h(x)dx.$$

Examples

Find the integrals of the following functions:

1. $f(x) = 2x$.
2. $f(t) = 3t^2 + t + 1$.
3. $f(x) = \frac{1}{2}x^{-3/4} + \frac{1}{x^2} - \sqrt{x}$.

Answer:

1. $\int 2xdx = 2\int xdx = x^2 + c$ because $\dfrac{d}{dx}(x^2 + c) = 2x$.

   Hence, $2x$ is the *derivative* of $x^2 + c$, and $x^2 + c$ is the *antiderivative* of $2x$.

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2. $\int(3t^2 + t + 1)dt = t^3 + \frac{1}{2}t^2 + t + c$
3. $\int(\frac{1}{2}x^{-3/4} + \frac{1}{x^2} - \sqrt{x})dx = 2x^{1/4} - x^{-1} - \frac{2}{3}x^{3/2} + c.$

Example

Calculate:

1. $\int_0^1 (3t^2 + t + 1)$.
2. $\int_\pi^x \sin t dt$.

Answers

1. $\int_0^1 (3t^2 + t + 1) = [t^2 + \frac{1}{2}t^2 + t]_0^1 = 2.5$

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2. $\int_\pi^x \sin t dt = -[\cos t]_\pi^x = -\cos x - 1$.

Calculate

1. $\int \sqrt{1+2t}dt$.
2. $\int_{1-3x}^{1} \frac{u^3}{1+u^2} du$.

Answer:

1. Let $v = 1 + 2t \Rightarrow \dfrac{dv}{dt} = 2 \Rightarrow \dfrac{dv}{2} = dt$. Hence,

$$\int \sqrt{1+2t}dt = \frac{1}{2}\int \sqrt{v}dv = \frac{1}{3}v^{3/2} + c = \frac{1}{3}(1+2t)^{3/2} + c.$$

(Integration by substitution)

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2.

$$\begin{aligned}\int_{1-3x}^{1} \frac{u^3}{1+u^2}du &= \int_{1-3x}^{1}(u - \frac{u}{1+u^2})du \\ &= \frac{1}{2}[u^2 - \ln(1+u^2)]_{1-3x}^{1} \\ &= \frac{1}{2}(1-\ln 2) - \frac{1}{2}((1-3x)^2 - \ln(1+(1-3x)^2).\end{aligned}$$

Application

The MC and MR of a company are given by $2Q - 4$ and $200 - 4Q$ respectively. Find the company's TC and TR functions. Note, $TR = 0$ and $TC = 100$ when $Q = 0$.

$$MC = \frac{dTC}{dQ} = 2Q - 4$$
$$dTC = (2Q - 4)dQ$$
$$\int dTC = \int (2Q - 4)dQ$$
$$TC = Q^2 - 4Q + c.$$

When $Q = 0$, $TC = 100$. Hence,

$$TC = Q^2 - 4Q + 100.$$

$$MR = \frac{dTR}{dQ} = 200 - 4Q$$

$$TR = \int (200 - 4Q)dQ = 200Q - 2Q^2 + c$$

When $Q = 0$, $TR = 0$. Hence,

$$TR = 200Q - 2Q^2.$$

A particle moves along a line so that its velocity at time t is

$$v(t) = t^2 - t - 6.$$

1. Find the displacement of the particle during the time period $1 \leq t \leq 4$.
2. Find the distance traveled during the time period.

1.

$$s(4) - s(1) = \int_1^4 v(t)dt = \int_1^4 (t^2 - t - 6)dt = -4.5.$$

This means that the particle moved $4.5m$ towards left.

2. $v(t) = t^2 - t - 6 = (t-3)(t+2)$ and so $v(t) \leq 0$ on the interval $[1, 3]$ and $v(t) \geq 0$ on the interval $[3, 4]$.

Total distance

$$\begin{aligned}\int_1^4 |v(t)|dt &= \int_1^3 [-v(t)]dt + \int_3^4 v(t)dt \\ &= \int_1^3 (t^2 - t - 6)dt + \int_3^4 (t^2 - t - 6)dt \\ &= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4 \\ &= 61/6 = 10.17m.\end{aligned}$$

Recommended Book: **Calculus By James Stewart.**

To book a maths/stats appointment...

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