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Network Games, Peer Effect and Neutral Transfers

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Abstract

We study properties of collective action problems bounded by minimal contributions as well as endowment and variable contributions are neighbourhood dependent. We relate nearness to non-interior agents and its implication for interior contribution. Here, we see the aspects of node distance to non-interior agents which have implications for interior agents. Endowments may be redistributed among agents. We highlight strict conditions for budget-balanced transfers for which neighbourhood contributions and individual residual consumption are invariant. Agents may or may not be concerned about neighbourhood outcomes. We find that welfare is self-correcting and neither cases are relevant to the overall welfare impact of neutral transfers.

JEL classification: C72, D85, H41.

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1 Introduction

The study of public good provision through contribution from private agents is important in that they provide us with roadmaps in which regulatory and distributive policies can be used to achieve optimal outcomes. In recent times there has been increased focus on strategic interactions between agents involved in actions that may lead to a collective benefit. We see a representation of this behaviour in the model to which economic agents make trade off decisions between private consumption and public good provision as introduced in the seminal work of Bergstrom, Blume, and Varian (1986). Such a problem has since then, been further analysed from a unique network of social interactions mainly by Allouch (2015). Hence, interactions are then linked directly to neighbourhood interactions such that agents only benefit directly from local public good provision (neighbourhood collective effort). This idea implicitly defines agents preference as those arising ultimately from neighbourhood contribution and private consumption. Hence, the direct benefit arises from the basic need to achieve a certain level of neighbourhood provision alone. The product is an equilibrium arising from substitution relationship[1] with generic public good characteristics such as free riding and active contribution.

For this work, we explore properties of a collective action game where actions of economic agents represent contributions to a local(neighbourhood) public good[2][3], as is similarly found within the framework of Bergstrom et al. (1986). However, modify aspects of the contribution so that the total contribution of each agent is made up of a fixed element[4] and the variable portion implying the amount that varies according on their mean-neighbours contributions. The utilisation of mean-neighbours as well as mean-neighbourhood contribution can be traced to the payoff introduced in the article by Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2010) where the *Example 3* maps the payoff of an agent to its neighbours average action. As such, we specify a best reply which is a game of strategic compliment and as such, contributions are a positive function

[1]Games with linear best replies that reveal strategic substitution are as well discussed in Bramoullé and Kranton (2007) and/or Bramoullé, Kranton, and D'amours (2014).

[2]The neighbourhood of an agent is defined as a group made up strictly of an agent and his neighbours, i.e, all those the agent is directly connected to.

[3]The term *Public Good* for this paper is the aggregate neighbourhood contribution.

[4]That is the portion which is independent of the contributions of other agents.

the mean-neighbours contributions. A quality of this best reply is that it is indicative of a system where an agents' payoff may or may not be a function of their neighbourhood public good. Primarily, the indirect utility function reveals agents who are interested in making contributions that exceed the agents' mean-neighbours contribution. A reason for this might be linked to higher-order preferences in which issues to do with reputation, prestige, and other needs which transcend simply achieving the required overall contributions. Strategic behaviors arising from the game are complementary in nature. A characteristic that is less common among works on private contributions to a public good. Nash equilibrium is uniquely defined under Ballester, Calvó-Armengol, and Zenou (2006) and a simple algorithm for computation are shown.

We discuss qualities of interconnections for which close attention is given to the nearness of an agent to less-endowed agents and its implications for potentially wealthy agents. We observe the importance of network concepts including shortest paths on the overall contribution an agent makes given the presence of non-interior agents. One major part we find here is that being close to a non-interior agent, though relevant, becomes more useful depending on the neighbourhood population of each agent within the pathway to the non-interior agents. In doing this, we explore the role of contagion through an undirected network as a different way to analyse peer effect in networks. The flow, to an extent, may be likened to the fictitious default sequence (as in Eisenberg and Noe (2001)) except for several divergences which include the movement of default which may or may not result in cyclical defaults.

In the last part, we pay brief attention to redistributive policies. We distinguish equilibrium contribution into rich agents, (whose equilibrium contributions are interior to their endowment), and poor agents who contribute their full endowment. As such, we observe the impact of wealth transfer among rich agents as well as conditions for the famed *transfer neutrality* as in Bergstrom et al. (1986) to hold. with utilitarian welfare, We then go further to access the impact of neutral transfers among rich agents on overall welfare. Agents whose wealth is constant given a transfer policy face no change in their payoff. Also, we find that neutral policies achieve the same overall welfare regardless of the fact agents may or may benefit directly from neighbourhood contributions. Lastly, we describe conditions for welfare improvement due to neutral pairwise transfers when private consumption is also beneficial to agents.

1.1 Related Literature

As described earlier, our model builds on specifically from the Bergstrom et al. (1986) framework which has been well studied within the economic network framework in works including Allouch (2015), Allouch and King (2021), Allouch (2017), Allouch and King (2018) (for substitute game with upper limits), Bourlès, Bramoullé, and Perez-Richet (2017)(where agents care about the payoff of neighbours), etc. Also, well linked to our best replies are those that may arise from contribution games as introduced mildly in Galeotti et al. (2010). Games of strategic complements in undirected networks include Belhaj, Bervoets, and Deroïan (2016), Belhaj and Deroïan (2019), Belhaj, Bramoullé, and Deroïan (2014), Galeotti, Golub, and Goyal (2020), Ballester et al. (2006). For directed networks, we have mainly those involved in financial contagion and systemic risk including Eisenberg and Noe (2001), Demange (2016), Feinstein (2017), to mention but a few.

For our main focus, we discuss neutrality in pairwise transfers between agents. This concept has been well introduced in Bergstrom et al. (1986) and discussed in Allouch (2015), Allouch (2017), and also discussed in specific instances such as when agents are altruistic as found in Bourlès et al. (2017).

2 The Model

Assume an economy with $\mathcal{N} = \{1, \ldots, n\}$ set of agents. Let $\mathcal{N}_i \subset \mathcal{N}$ be the set of neighbours of an agent $i \in \mathcal{N}$ and $n_i = |\mathcal{N}_i|$ as the cardinality of the set. Each agent makes a decision based on their limited endowment given as w_i. The agent contributes a fraction of w_i to a create a local public good and consumes what is left over. However, let the agent's preference mainly depend on the combination of their total neighbourhood contribution and consumption utility. The consumption utility of the agent is in turn a function of their contribution relative to their entire neighbours.

The Agents' Payoff

Let then q_i represents agent i's contribution while Q_i his total neighbourhood contribution. Also, let x_i be the amount which agent i consumes given q_i so that we have the following budget equation for the agent i as $x_i + q_i = w_i$. We then have agent i's preference given as $U_i(q_i, Q_{-i})$ such that $Q_i = q_i + Q_{-i}$ so that $U_i(q_i, Q_{-i})$ is twice differentiable in q_i. We adopt a Linear quadratic utility form[5] as shown below;

$$U_i(q_i, Q_{-i}) = a\frac{n_i q_i}{Q_{-i}} w_i + (1-a)Q_i - 0.5\frac{n_i}{Q_{-i}} q_i^2. \tag{2.1}$$

The parameter $(1-a) \in]0,1[$ is a constant and as such represents the degree of neighbourhood complementary relationship arising from the degree to which local public good is of benefit to the agent. This leaves the constant $a \in]0,1[$ as a residual for which measures the proportion to which an agent i benefits from their contribution as a function of their endowment. Depending on the magnitude of this constant, we could arrive at even deeper intuition. An example of this is observed in the later part where we assume $a = 0.5$ which then makes the agent i's payoff as a direct function of their private consumption x_i. The payoff $U_i(q_i, Q_{-i})$ for each agent $i \in \mathcal{N}$ generates as its FOC i.e., $U_i^{'}(q_i)$, a linear best reply function as;

$$q_i^{br} = \min\left\{aw_i + (1-a)n_i^{-1}Q_{-i}, w_i\right\}. \tag{2.2}$$

From (2.1), we see that agent i directly benefits from a fraction $(1-a)$ of their neighbourhood contributions. However, it is observable that in cases where the agent is only interested in their individual contribution such that $(1-a)q_i$ replaces $(1-a)Q_i$ in the payoff function, or best reply in eq. (2.2) remains unchanged. There are possible reasons why agents may be uninterested in their local public good provision. Take for example, a business organisation where work-hours put in are seen as a sign of dedication and rewarded as opposed to productivity. A worker who then puts in overtime work is treated with better reception compared to one who spends the normal working hours but accomplishes more.

Moving on, we introduce an endowment based restriction as follows;

Assumption 1. *For each agent $i \in \mathcal{N}$, $\mathcal{N}_i \neq \{\}$ and $w_i > 0$.*

[5] As is widely use in complementary games for example, (Belhaj & Deroïan, 2019).

While the standard public good problem in Networks shows strategic substitutes[6], it is possible to find complements arising from collective action. This may be due to reputation or prestige arising from the total donations such that an agents' preferences transcend the basic need of aggregate neighbourhood provision of a public good. As stated earlier, we see this concept introduced well in Glazer and Konrad (1996) as well as hinted considerably in Fehr and Schmidt (1999).

Note that from (2.2), the PSNE Nash equilibrium vector $\mathbf{q}^*(\boldsymbol{G}, \boldsymbol{A})$ is uniquely defined in so far $a > 0$, i.e., $a \in]0, 1[$, which always holds[7] given the spectral properties of the network. To find the Nash Equilibrium, the following algorithm is proposed;

1. Let $\kappa \in K$ for $K = \{0, 1, \ldots, \kappa, \kappa + 1, \ldots, \}$ be number of iteration needed to find the equilibrium.

2. Begin by assuming that for each $i \in \mathcal{N}$, $q_{j\in\mathcal{N}_i} = w_{j\in\mathcal{N}_i}$ so that $Q_{-i}(0) = \sum_{j\in\mathcal{N}_i} w_j$.

3. Then solve for each $q_i(0)$ where;

$$q_i(0) = aw_i + (1-a)\frac{Q_{-i}(0)}{n_i} = aw_i + (1-a)\frac{\sum_{j\in\mathcal{N}_i} w_j}{n_i}$$

   .

4. If any $q_i(0) > w_i$, then $q_i(1) = w_i$. Otherwise, $q_i(1) = q_i(0)$.

5. Solve for $q_i(\kappa) = aw_i + (1-a)\frac{Q_{-i}(\kappa)}{n_i}$ until $q_i(\kappa + 1) = q_i(\kappa)$ for all agent $i \in \mathcal{N}$.

6. Then $q_i(\kappa)$ such that $q_i(\kappa + 1) = q_i(\kappa)$ holds is the PSNE contribution q_i^* $\forall\ i \in \mathcal{N}$.

7. The sequence terminates.

The sequence above is draws inspiration from network games of strategic complementarity with linear best replies. Most pertinent to the sequence above is the *Fictitious Default Sequence* proposed by Eisenberg and Noe (2001). This is such that $q_i(\kappa)$ is non-increasing in κ for $\kappa \in K$ meaning that contributions either decrease or remain static at each iteration.

[6]Examples include Allouch (2015), Bramoullé et al. (2014), Allouch (2017), etc.

[7]This is the case as it is only then we have $v_{max}(\boldsymbol{AG}) < 1$ which guarantees uniqueness in complementary games. See (Ballester et al., 2006) for formal proof.

Network

Let $\boldsymbol{G} = (g_{ij})_{j\in\mathcal{N}_i} \in \{0,1\}^{n\times n}$ represent the adjacency matrix whose elements such that $g_{ij} = g_{ji} = 1$ if $j \in \mathcal{N}_i$ and $g_{ij} = 0$ otherwise. Let $\nu_1(\boldsymbol{G}), \ldots, \nu_n(\boldsymbol{G})$ be the eigenvalues of the matrix $(\boldsymbol{G})$. Also, we introduce a diagonal matrix $\boldsymbol{A} = \big((1-a)\cdot n_i^{-1}\big)_{i\in\mathcal{N}} \in \mathbb{R}^{n\times n}$. Also we have the vector $\boldsymbol{Q} = (\boldsymbol{I}+\boldsymbol{G})\cdot \mathbf{q}$ for $\boldsymbol{Q} = (Q_i)_{i\in\mathcal{N}} \in \mathbb{R}^n_+$ represents the vector of aggregate neighbourhood contribution. Let $\mathbf{q}^* = (q_i^*)_{i\in\mathcal{N}} \in \mathbb{R}^n_+$ represent the Pure Strategy Nash Equilibrium (PSNE) vector, $\boldsymbol{w} = (w_i)_{i\in\mathcal{N}} \in \mathbb{R}^n_+$ representing the column vector of half of each agent i endowment.

3 Endowment and Interior Contribution

In this part, we look at network properties as well as their ability to shape the level of contribution an agent potentially makes. We mildly discuss instances that may enable an agent to contribute below its endowment and the role contagion and cascade of the effect agents with less endowment play in such roles. Observe the following remark;

Remark 3.1. *Say for each agent $i \in \mathcal{N}$, $q_i^* \leq w_i$.*

This then means that the PSNE is given in weighted Bonacich centrality vector as;

$$\mathbf{q}^*(\boldsymbol{G},\boldsymbol{A}) = a(\boldsymbol{I} - \boldsymbol{A}\boldsymbol{G})^{-1}\cdot \boldsymbol{w} \tag{3.1}$$

Our aim in this section is to observe if indeed it is possible to have (3.1) for the set $\mathcal{N}$ and also to understand and predict the magnitude of interior contribution based on the *Shortest Path* concept. With this we write the following results;

Lemma 1. *Let the Unweighted Bonacich centrality be $\boldsymbol{B}(\boldsymbol{G},\boldsymbol{A}) = (\boldsymbol{I}-\boldsymbol{A}\boldsymbol{G})^{-1}\cdot\mathbf{1}$ such that $\boldsymbol{B}(\boldsymbol{G},\boldsymbol{A}) = (\beta_i)_{i\in\mathcal{N}} \in \mathbb{R}^n_+$. Then unweighted Bonacich centrality is identical for all agent $i \in \mathcal{N}$, i.e., $\boldsymbol{B}(\boldsymbol{G},\boldsymbol{A}) = a^{-1}\cdot\mathbf{1}$.*

Proof. Since $\boldsymbol{AG} \cdot \mathbf{1} = (1-a) \cdot \mathbf{1}$, then it means for each $i - th$ element in $\boldsymbol{B}(\boldsymbol{G}, \boldsymbol{A})$,

$$\begin{aligned}\beta_i(\boldsymbol{G}, \boldsymbol{A}) &= 1 + (1-a)\left(n_i^{-1}\sum_{j\in\mathcal{N}_i} g_{ij}\right) + \left((1-a)\left(n_i^{-1}\sum_{j\in\mathcal{N}_i} g_{ij}\right)\right)^2 + \dots \\ &= 1 + (1-a) + (1-a)^2 + \dots \\ &= \frac{1}{a}.\end{aligned}$$

■

Also, we have the following property of the equilibrium which ranks agents endowment and highlight its relevance to the interior nature of the PSNE as given below;

Lemma 2. *If we have $\boldsymbol{w} = w \cdot \mathbf{1}$ so that wealth is identical for all agent $i \in \mathcal{N}$. Also, PSNE becomes simply the endowment of each agent i i.e., $\mathbf{q}^*(\boldsymbol{G}, \boldsymbol{A}) = \boldsymbol{w}$.*

Proof. Intuitive since we have the weighted Bonacich centrality (PSNE) as, $\mathbf{q}^*(\boldsymbol{G}, \boldsymbol{A}) = (\boldsymbol{I} - \boldsymbol{AG})^{-1} \cdot a \cdot w \cdot \mathbf{1} = a \cdot w \cdot (\boldsymbol{I} - \boldsymbol{AG})^{-1} \cdot \mathbf{1} = a \cdot w \cdot a^{-1} \cdot \mathbf{1} = \boldsymbol{w}$. ■

For the sake of our analysis further, we begin with the following definition:

Definition 1. *An agent $i \in \mathcal{R}$ with $w_i \in \mathbb{R}_{++}$ is described as;*

1. A rich agent: *if $q_i^* < w_i$, and*
2. A poor agent: *if $q_i^* = w_i$.*

Linking the definition above to lemma 2 leads us to the interpretation that where agent have equal endowment, all agents are poor agents. We then specify sufficient conditions for PSNE which includes at least one rich agent in the statement below;

Theorem 1. *Given assumption 1, if an agent $i \in \mathcal{N}$ such that $q_i^* < w_i$ exist at PSNE, then $\exists$ an agent $j \in \mathcal{N}$ and $j \neq i$ such that $q_j(Q_{-j}) \geq w_j$ implying $q_j^* = w_j$.*

Proof. Given $b = (1-a)$, we have the best reply given as

$$q_i^{br} = \max\left\{aw_i + bn_i^{-1}Q_{-i}, w_i\right\}.$$

Since links are bilateral, then assume an agent i such that $q_i^* < w_i$. This means at least one $j \in \mathcal{N}_i$ is such that $w_j < w_i$ since it has to be the case that $n_i^{-1} w_{-i \in \mathcal{N}_i} < w_i$. In turn, agent j can only have $q_j^* < w_j$ if and only if at least an agent $k \in \mathcal{N}_j$ for $k \neq i$ is such that $w_j > w_k$. Thus the sequence $w_i > w_j > w_k$. This implies that there exists at least one agent l such that no agent $m \in \mathcal{N}_l$ is such that $w_l > w_m$. ■

This means that for each $\mathcal{N}$ set of agents, at $\mathbf{q}^*$, depending on the magnitude of b, there may or may not $\exists$ at least one agent, say an agent $k \in \mathcal{N}$ such that it is described as a *poor agent*.

The summary of the intuition behind theorem 1 above is to point out the property of our contribution game which is that either all agents are maximally contributing or at least one agent is so that it is not possible to have a PSNE that no agent is contributing maximally. This instance of maximal contribution is when $\boldsymbol{w} = w \cdot \mathbf{1}$ so that endowments are identical. Otherwise, an agent $i \in \mathcal{N}$ who has the least endowment will always contribute maximally i.e, $q_i^* = w_i$.

Another intuition is the peer effect whereby agents who maximally contribute are likely connected to agents with lower endowment such that those with lower endowment are forced into maximal contribution while such maximal contributions are low enough to achieve a contribution lower than the endowment of a direct (as well as indirect agents depending) connected agent. For rich agents, it is possible to understand the magnitude of contribution by tracking its proximity to a poor agent. We introduce the following definitions;

Definition 2 (Shortest path)**.** *Assume a subset $\{i, j\} \subset \mathcal{N}$ such that $g_{kl} = g_{lk} = 1$. Let us have $s \in \mathcal{S}(i, j)$ for which $|\mathcal{S}(i, j)| \in \mathbb{N}_+$ representing the set of all possible paths between agent i and agent j who are non-adjacent. Then $\forall$ $\{i, j\} \in \mathcal{N}$, the* Weak Shortest Path (WSP) *$\mathcal{P}_t(i, j) = (1, 2, \dots \theta)$, where $1 = i$ and $\theta = j$, is the path for which*

$$\sum_{i=1}^{\theta-1} g_{i,i+1} | \mathcal{P}_t(i, j) \leq \sum_{i=1}^{\theta-1} g_{i,i+1} | \mathcal{P}_{-t \neq t}(i, j),$$

where $\sum_{i=1}^{\theta-1} g_{i,i+1} | \mathcal{P}_{t \in \mathcal{S}(i,j)}(i, j) \in \mathbb{N}_+$. If $\sum_{i=1}^{\theta-1} g_{i,i+1} | \mathcal{P}_t(i, j) \lneq \sum_{i=1}^{\theta-1} g_{i,i+1} | \mathcal{P}_{-t \neq t}(i, j)$ then $\mathcal{P}_t(i, j)$ is a Strict Shortest path (SSP)*.*

Definition 3 (Shortest Weighted Path)**.** *Given definition 2,* $\forall$ $\{i, j\} \in \mathcal{N}$*, the* Weak Shortest Weighted Path (WSWP) $\mathcal{P}_s(i, j) = (1, 2, \ldots \theta)$*, where* $1 = i$ *and* $\theta = j$*, is the path for which*

$$\sum_{i=1}^{\theta-1} n_i^{-1} g_{i,i+1} | \mathcal{P}_s(i, j) \leq \sum_{i=1}^{\theta-1} n_i^{-1} g_{i,i+1} | \mathcal{P}_{-s \neq s}(i, j),$$

Where $\sum_{i=1}^{\theta-1} n_i^{-1} g_{i,i+1} | \mathcal{P}_{s \in \mathcal{S}(i,j)}(i, j) \in \mathbb{R}_+$*. If* $\sum_{i=1}^{\theta-1} n_i^{-1} g_{i,i+1} | \mathcal{P}_s(i, j) \lneq \sum_{i=1}^{\theta-1} n_i^{-1} g_{i,i+1} | \mathcal{P}_{-s \leq s}(i, j)$ *then* $\mathcal{P}_s(i, j)$ *is a* Strict Shortest Weighted Path (SSWP).

Assumption 2. *For every* $\{i, j\} \in \mathcal{N}$*, there exits a* $\mathcal{P}(i, j) \in \mathbb{R}_+$*.*

One direct implication of this assumption is the fact that for each $\{i, j\} \in \mathcal{N}$, there exists a shortest path $\mathcal{P}_t(i, j)$ as well as a Shortest weighted path (SWP) $\mathcal{P}_s(i, j)$ for which both sets may not be equal to each other.

Lemma 3. *Assume a SSWP given as* $\mathcal{P}_s(i, j) = (1, 2 \ldots, \theta - 1, \theta)$ *for any* $\{i, j\} \in \mathcal{N}$ *and graph* $\boldsymbol{AG}$ *where* $i = 1$ *and* $j = \theta$*. Let* $\mathcal{P}^d(i, j)$ *be the duality of any path* $\mathcal{P}(i, j)$ *(which are always totally ordered). The following statement holds true;*

1. $n_i \geq 1$ *for* $i = 1, \theta$*.*

2. $n_k \gneq 1$ *for* $1 < k < \theta$*.*

3. $\mathcal{P}_s(j, i) = \mathcal{P}_s^d(i, j)$ *if and only if* $n_i = n_j$*.*

Proof. Since $i = 1$ and $j = \theta$ represents the start and endpoint of the shortest path $\mathcal{P}_s(i, j)$, then it is possible that $n_i = 1$ given that 2 is the only adjacent agent to agent i. The same goes for agent θ who may be adjacent strictly to agent $\theta - 1$. However, an agent $1 < k < \theta$ means that agent k is in-between at least 2 adjacent agents for the sake of $\mathcal{P}_s(i, j)$. Hence, it is a contradiction that $n_k < 2$.

We recall the shortest path from agent i to agent j as $\mathcal{P}_s(i, j) = (1, 2, \ldots, \theta - 1, \theta)$. Let us have any path from agent j to agent i $\mathcal{P}(j, i) = (1, 2, \ldots, \hat{\theta} - 1, \hat{\theta})$ in which $1 = j$ and $i = \hat{\theta}$. Then for $\mathcal{P}_s(j, i) = (\theta, \theta - 1, \ldots, 2, 1)$, it has to be that $\sum_{j=\theta}^{2} n_j^{-1} g_{j,j-1} | \mathcal{P}_s^d(i, j) \leq \sum_{j=1}^{\hat{\theta}-1} n_j^{-1} g_{j,j+1} | \mathcal{P}_{-s \neq s}(j, i)$. Let us also have $\varrho_s(i, j) = (g_{1,2}, g_{2,3}, \ldots, g_{\theta-2,\theta-1}, g_{\theta-1,\theta})$ be the set of edges/links corresponding to the shortest path $\mathcal{P}_s(i,j)$ then it is the case that

$$\begin{aligned}\varrho_s(i,j) &= \left(\frac{1}{n_1}g_{1,2}, \frac{1}{n_2}g_{2,3}, \ldots, \frac{1}{n_{\theta-2}}g_{\theta-2,\theta-1}, \frac{1}{n_{\theta-1}}g_{\theta-1,\theta}\right)\\ &= \left(\frac{1}{n_1}, \frac{1}{n_2}, \ldots, \frac{1}{n_{\theta-2}}, \frac{1}{n_{\theta-1}}\right),\end{aligned}$$

since all existing $g_{ij} = g_{ji} = 1$. Then the set of edges corresponding to $\mathcal{P}_s^d(i,j)$ becomes

$$\varrho_s^d(i,j) = \left(\frac{1}{n_\theta}, \frac{1}{n_{\theta-1}}, \ldots, \frac{1}{n_3}, \frac{1}{n_2}\right).$$

Then by process of elimination, the set difference is given by,

$$\varrho_s(i,j) - \varrho_s^d(i,j) = \frac{1}{n_1} - \frac{1}{n_\theta}.$$

This means that $\varrho_s(i,j) - \varrho_s^d(i,j) = \{\}$ if and only if $n_i = n_j$ since $i = 1$ and $j = \theta$.

■

Definition 4 (Shortest Walk). *$\forall \{i,j\} \in \mathcal{N}$ and a path $\mathcal{P}_t(i,j) = (1, 2, \ldots \theta)$ for $t \in \mathcal{S}(i,j)$ (which is the WSP), the* Shortest Walk *between agent i and agent j denoted by $\omega_s(i,j)$ is simply given defined as follows*

$$\omega_t(i,j) \stackrel{\text{def}}{=} (1-a)^{\theta-1} \prod_{i=1}^{\theta-1} n_i^{-1} g_{i,i+1} | \mathcal{P}_t(i,j). \tag{3.2}$$

Lemma 4. *Given the graph $\boldsymbol{AG}$, for any $\{i,j\} \in \mathcal{N}$ we have that for $\omega_t(i,j)|\mathcal{P}_t(i,j)$, $\mathcal{P}_t(i,j)$ is not necessarily the shortest path $\mathcal{P}_s(i,j)$.*

Proof. The principle below shows the proof WLOG: Assume $a, b, c, , d \in]0,1[$ then say we know conclusively that $a + b \lesseqgtr d + c$, we are not able to conclude as to the magnitude of $a * b$ when compared to $d * c$. ■

Definition 5. *Given $\{i,j\} \in \mathcal{N}$, the shortest path $\mathcal{P}_{t\in\mathcal{S}}(i,j)$ is defined as a Relevant Shortest part if and only if there exists no agent k such that $k \in \mathcal{P}_t(i,j)$ and $q_k^* = w_k$.*

Lemma 5. *Given $\{i,j\} \in \mathcal{N}$, assuming between the shortest path $\mathcal{P}_t(i,j)$ we have a $k \in \mathcal{P}_t(i,j)$ such that $q_k^* = w_k$, then $\mathcal{P}_t(i,j)$ is no longer a* Relevant Shortest Path. *As such the shortest path becomes either of the following;*

1. $\mathcal{P}_{\hat{t}}(i,j)$ *such that* $k \notin \mathcal{P}_{\hat{t}}(i,j)$ *and* $\sum_{i=1}^{\theta-1} g_{i,i+1}|\mathcal{P}_{\hat{t}}(i,j) = \sum_{i=1}^{\theta-1} g_{i,i+1}|\mathcal{P}_t(i,j)$ *where* $\mathcal{P}_t(i,j)$ *is a weak shortest path and;*

2. $\sum_{i=1}^{\theta-1} g_{i,i+1}|\mathcal{P}_{\hat{t}}(i,j) > \sum_{i=1}^{\theta-1} g_{i,i+1}|\mathcal{P}_t(i,j)$ *but* $\sum_{i=1}^{\theta-1} g_{i,i+1}|\mathcal{P}_{\hat{t}}(i,j) \leq \sum_{i=1}^{\theta-1} g_{i,i+1}|\mathcal{P}_{-\hat{t}\neq t\neq s}(i,j)$ *in the case where* $\mathcal{P}_t(i,j)$ *is the strict shortest path.*

Proof. This is intuitive given the strategic dominance of q_k^*. ■

This describes our shortest path as simply the path between agent i and agent j that has the least amount of edges/link $\left(n_i^{-1} g_{i,i+1}\right)$'s in between them. The importance of this property lies in understanding the link between closeness to a poor agent(s) and interior contribution. Additionally, our assumption 2 implies that the network graph given by the adjacency matrix $\boldsymbol{G}$ is such that there is no disjointed sub-network, and an undirected walk exits between every 2 agents in the network. We summarise in the following results;

Theorem 2. *Assuming an agent* j *such that PSNE is its endowment, i.e.,* $q_j^* = w_j$*, then assume for* $\{j\}^c \subset \mathcal{N}$ *such that* $\{j\}^c \cap \{j\} = \{\}$*,* $w_i = w$ *for all* $i \in \{j\}^c$*. Then for any* $i, k \in \{j\}^c$ *for which* $|\mathcal{P}_t(i,j)| = |\mathcal{P}_t(k,j)|$ *(where* $\mathcal{P}_t(\cdot)$ *is the SSP) and* $\omega_t(i,j) = \omega_t(k,j)$*, the PSNE* $q_i^* \geq q_k^*$ *so far as* $n_i \geq n_j$*.*

Proof. Given agent j has the PSNE $q_j^* = w_j$ while $\mathcal{P}_t(i,j) = (1, 2 \ldots, \theta-1, \theta)$ represents the path yielding the shortest walk from agent i to agent j. Then since excluding w_j, the equilibrium would simply have been $q_i = aw + \frac{(1-a)(n_{.i})}{n_i} w = w$. Assuming then that $\kappa(\theta-1) \in K$ representing the iteration akin to the number of edges of the shortest path for $\theta = |\mathcal{P}_t(i,j)| \in \mathbb{N}_+$, then we have the value of $q_i(\kappa)$ as follows;

$$
\begin{aligned}
q_i(k, w_j) &= aw\left(1 + \sum_{\iota=1}^{\theta-2} \frac{(1-a)^\iota}{\prod_{i=1}^{\iota} n_i}\right) + w\left(\sum_{\iota=1}^{\theta-1} \frac{(1-a)^\iota (n_\iota - 1)}{\prod_{i=1}^{\iota} n_i}\right) + \frac{(1-a)^{\theta-1}}{\prod_{i=1}^{\theta-1} n_i} w_j \\
&= w\left(a + \sum_{\iota=1}^{\theta-2} \frac{(1-a)^\iota (n_\iota - 1 + a)}{\prod_{i=1}^{\iota} n_i} + (n_{\theta-1} - 1)\frac{(1-a)^{\theta-1}}{\prod_{i=1}^{\theta-1} n_i}\right) + \frac{(1-a)^{\theta-1}}{\prod_{i=1}^{\theta-1} n_i} w_j \\
&= w\left(a + \sum_{\iota=1}^{\theta-2} \frac{(1-a)^\iota (n_\iota - 1 + a)}{\prod_{i=1}^{\iota} n_i} + (n_{\theta-1} - 1)\omega_t(i,j)\right) + \omega_t(i,j) w_j.
\end{aligned}
$$

Since it is the case that $\omega_t(i,j) = (1-a)^{\theta-1}\prod_{i=1}^{\theta-1} n_i^{-1}$. We also know that the value $(1-a)^{\theta-1}$ is the same for either agent i and agent k since $|\mathcal{P}_t(i,j)| = |\mathcal{P}_t(k,j)|$. Then given $\prod_{i=1}^{\theta-1} n_i = \prod_{k=1}^{\theta-1} n_k$ and the value $(1-a) < 1$, we have the most significant amount in the bracket above as $\frac{(1-a)(n_i-1)}{n_i}$. This is because as $i \to \theta - 2$, then even if $\prod_{i=1}^{\theta-1} n_i$ becomes larger due to some added n_l, then its impact on the numerator $\frac{n_l-1}{n_l} \to 1$ the larger n_l is. But then we still have $\frac{\prod_{i=1}^{\theta-1} n_i}{n_l}$ on the numerator which is significantly whilst excluding the discount from $(1-a)^l$ which makes such value trivial. Since then $\frac{(1-a)(n_i-1)}{n_i} \to (1-a)$ the larger n_i becomes, our theorem holds.

■

Our main take from this statement would be that while it is easily deduced that agents who are closest (or have fewer walks) to a poor agent are more likely contributing less due to such proximity, the neighbourhood composition of 2 agents who might be equidistant to the poor agent may still lead to disparity in the level of contribution. This links to the concept of closeness centrality but it is quickly observable that such centrality accounts for the shortest path to every node as opposed to simply a poor agent. We use the example below to illustrate further;

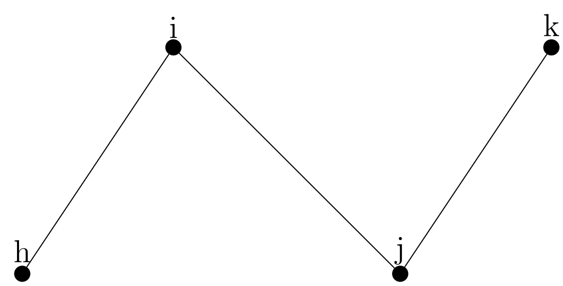


Figure 1: Line Network.

Example 1. Assume the network in fig. 1 where $\boldsymbol{w} = (12, 6, 15, 5)^\top$ and $a = \frac{1}{2}$. We have the

$$\mathbf{q}^* = (9, 6, 10.25, 5)^\top.$$

If on the other hand from fig. 1, we have $\boldsymbol{w} = (12, 12, 12, 5)^\top$, then

$$\mathbf{q}^* = (11.7, 11.5, 10.1, 5)^\top$$

.

In the example above, we see the relationship between the agent's contribution, wealth, and proximity to the interior(rich) or non-interior (poor) contributing agents. From the 2 examples, we see an instance where the presence of a least-endowed agent leads to interior contribution in only its direct neighbour and the second instance where all other agents contribute below the agents' endowment. This is because in example 1 the agent i has such low endowment such that a lower average neighbour endowment $\left(0.5\sum_{j\in\mathcal{N}_i} q_j\right)$, which arises from agent j being directly connected to agent k, is still greater than w_i. Broadly, a poorer agent reduces contribution from interconnected agents and its effect diminishes by the number of walks. However, this reduction is halted if there exists another poor agent who is unaffected by the poorest agent indirect effect.

4 Rich Agents and Neutral Transfers

Going further, we aim to explore the policy properties of our equilibrium. More precisely, we investigate the possibility of neutral transfers and their implication, if any, on agents' welfare. We begin by excluding agents whose contributions are non-interior (poorer agents) as follows; Let $\mathcal{R} = \{1, \ldots, r\}$ for $\mathcal{R} \subsetneq \mathcal{N}$, $r = |\mathcal{R}| \in \mathbb{N}$ represent the set of wealthy agents who are such that for an agent $i \in \mathcal{R}$, $q_i^* < w_i$ and then $\mathcal{N}/\mathcal{R}$ be the poorer agents so that another agent $j \in \mathcal{N}/\mathcal{R}$ when $q_j^* < w_j$. Also, let $\boldsymbol{A}_{\mathcal{R}} = \left((1-a)n_i^{-1}\right)_{i\in\mathcal{R}}$ and $\boldsymbol{G}_{\mathcal{R}} = (g_{ij})_{j\in\{\mathcal{N}_i\cap\mathcal{R}\}}$ be the $r \times r$ matrices which represent both the normalising parameter and the interconnection between wealthy agents. Lastly, we have the vector $\hat{\boldsymbol{w}}_{\mathcal{R}} = (\hat{w}_i)_{i\in\mathcal{R}}$ which is such that for an agent $i \in \mathcal{R}$, if an agent $j \in \{\mathcal{N}_i \cap \mathcal{N}/\mathcal{R}\}$, then $\hat{w}_i = aw_i + (1-a)n_i^{-1}w_j$ implying it adds its share of profit from poor neighbours[8]. Our Nash equilibrium contribution is then written as follows;

$$\mathbf{q}_{\mathcal{R}}^* = a \cdot (\boldsymbol{I} - \boldsymbol{A}_{\mathcal{R}}\boldsymbol{G}_{\mathcal{R}})^{-1} \cdot \hat{\boldsymbol{w}}_{\mathcal{R}}. \tag{4.1}$$

with this, we can proceed to observe further properties. One possible question would be the concept of wealth transfer and neutrality. We recall from Bergstrom et al. (1986)

[8]This implies that if $\{\mathcal{N}_i \cap \mathcal{N}/\mathcal{R}\} = \{\}$, then $\hat{w}_i = w_i$.

as well as Allouch (2015) that for a small tax/subsidy which we denote for an agent $i \in \mathcal{R}$ as $t_i \in \mathbb{R}$ such tax is said to be neutral if,

$$(x_i^t, Q_i^t) = (x_i^*, Q_i^*)$$

which implies that the agent's private consumption and neighbourhood contributions remain unchanged. Some guiding results still apply to this problem which we write below;

Theorem 3 (Pairwise Transfers)**.** *WLOG, assume $\{i, j\} \subsetneq \mathcal{R}$. A budget balanced transfer between a set of agents given as t_i and t_j such that $t_i + t_j = 0$ is neutral if and only if it;*

1. *leaves the set $\mathcal{R}$ constant at equilibrium, i.e $\mathcal{R}(\boldsymbol{t}) = \mathcal{R}$.*

2. *the pair share the same neighbourhood, i.e*

$$\mathcal{N}_i \cup \{i\} = \mathcal{N}_j \cup \{j\}.$$

Proof. See Proof of Theorem 3 in Allouch (2015). ∎

One quick observation from this statement would be that it would be impossible to make such transfers in a network as in fig. 1 as agent h and agent j are not within the same neighbourhood[9]. We can see this illustrated in fig. 2 in which apart from transfers between agent i and agent j in fig. 2c, no other transfer is feasible since no other 2 agents in either the fig. 2a, fig. 2b or fig. 2d has the same neighbourhood components. Another implication of the theorem above is that $n_i = n_j$ and also $Q_i^* = Q_i^*$. We proceed to establish some vector property conditions. Let $\boldsymbol{x}_{\mathcal{R}} = (x_i^*)_{i \in \mathcal{R}} \in \mathbb{R}_+$ be the private consumption of each agent $i \in \mathcal{R}$, we further write a complementary result on the properties of the transfer vector $\boldsymbol{t}$ as follows;

Theorem 4. *A vector of transfer $\boldsymbol{t}$ is neutral if $\boldsymbol{t}$ is an eigenvector corresponding to $\nu(\boldsymbol{AG}) = 1 - a$.*

[9]We treat agent k as though it is poor, i.e., agent $k \in \mathcal{N}/\mathcal{R}$.

Proof. So we know from Allouch (2015) that since for each agent $i \in \mathcal{R}$,

$$q_i^t - q_i^* = \left(aw_i + (1-a)n_i^{-1}w_{j\in\{\mathcal{N}_i \cap \mathcal{N}/\mathcal{R}\}} + t_i - x_i^t\right) - \left(aw_i + (1-a)n_i^{-1}w_{j\in\{\mathcal{N}_i \cap \mathcal{N}/\mathcal{R}\}} - x_i^*\right),$$
$$= t_i - (x_i^t - x_i^*).$$

So then if $x_i^t - x_i^* = 0$, then we have that $q_i^t - q_i^* = t_i$ so that changes in local contributions becomes;

$$Q_i^t - Q_i^* = t_i + \sum_{j\in\mathcal{N}_i} t_j = 0,$$

which is according to our neighbourhood homogeneous condition. This means that we have in vector form that

$$\mathbf{q}^t - \mathbf{q}^* = a\left(\boldsymbol{I} - \boldsymbol{A}_{\mathcal{R}}\boldsymbol{G}_{\mathcal{R}}\right)^{-1} \cdot \boldsymbol{t}$$
$$= \boldsymbol{t}.$$

This implies that

$$\left(\boldsymbol{I} - \boldsymbol{A}_{\mathcal{R}}\boldsymbol{G}_{\mathcal{R}}\right) \cdot \boldsymbol{t} = a \cdot \boldsymbol{t},$$

and as such we have

$$\boldsymbol{t} - a \cdot \boldsymbol{t} - \boldsymbol{A}_{\mathcal{R}}\boldsymbol{G}_{\mathcal{R}} \cdot \boldsymbol{t} = \mathbf{0}$$

or then can be written as

$$(1-a) \cdot \boldsymbol{t} = \boldsymbol{AG} \cdot \boldsymbol{t}. \tag{4.2}$$

This implies that $\boldsymbol{t}$ is the eigenvector of $\boldsymbol{AG}$ corresponding to $\nu = 1 - a$ from our spectral radius properties.

Another way to arrive at this condition could be by solving for the private consumption differential given as

$$\boldsymbol{x}^t - \boldsymbol{x}^* = \Delta x(\boldsymbol{t}) = \mathbf{0}.$$

So we then have that

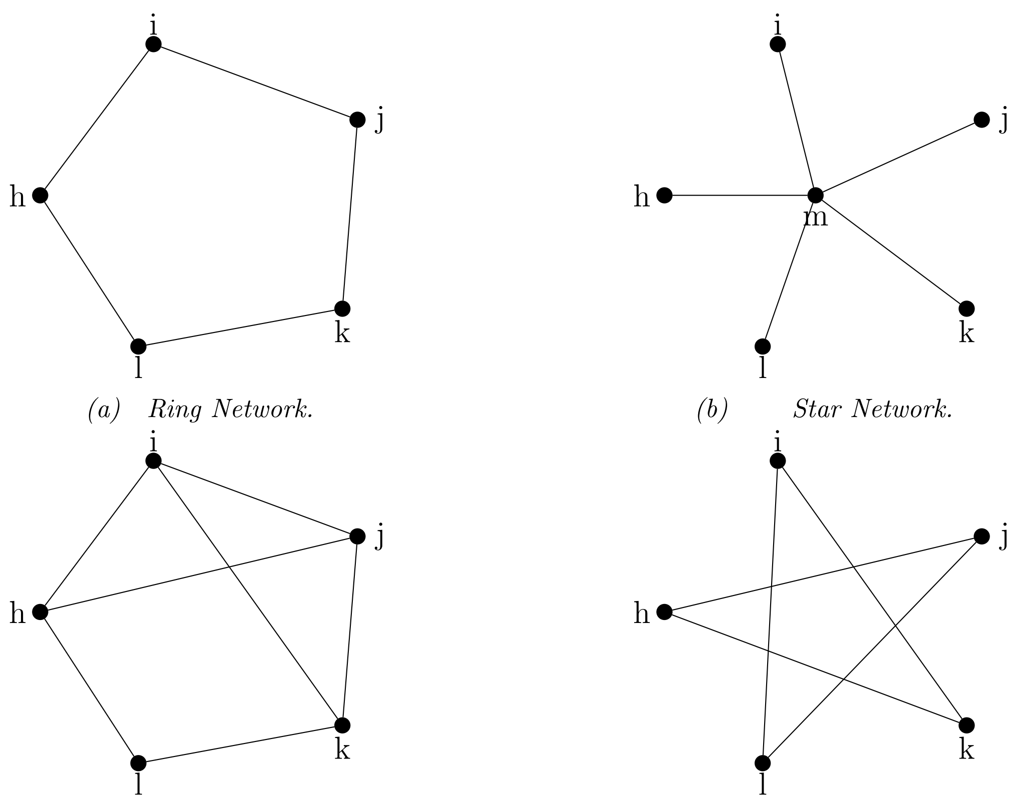
$$
\begin{aligned}
\Delta x(\boldsymbol{t}) &= \left(\boldsymbol{w}_{\mathcal{R}} + \boldsymbol{t} - a \cdot (\boldsymbol{I} - \boldsymbol{A}_{\mathcal{R}}\boldsymbol{G}_{\mathcal{R}})^{-1} \cdot (\hat{\boldsymbol{w}}_{\mathcal{R}} + \boldsymbol{t})\right) - \left(\boldsymbol{w}_{\mathcal{R}} - a \cdot (\boldsymbol{I} - \boldsymbol{A}_{\mathcal{R}}\boldsymbol{G}_{\mathcal{R}})^{-1} \cdot \hat{\boldsymbol{w}}_{\mathcal{R}}\right), \\
&= \boldsymbol{t} - a \cdot (\boldsymbol{I} - \boldsymbol{A}_{\mathcal{R}}\boldsymbol{G}_{\mathcal{R}})^{-1} \cdot (\hat{\boldsymbol{w}}_{\mathcal{R}} + \boldsymbol{t}) + a \cdot (\boldsymbol{I} - \boldsymbol{A}_{\mathcal{R}}\boldsymbol{G}_{\mathcal{R}})^{-1} \cdot \hat{\boldsymbol{w}}_{\mathcal{R}}, \\
&= \boldsymbol{t} - a \cdot (\boldsymbol{I} - \boldsymbol{A}_{\mathcal{R}}\boldsymbol{G}_{\mathcal{R}})^{-1} \cdot \boldsymbol{t} \\
&= \boldsymbol{0}
\end{aligned}
$$

which can then be written as;

$$
\boldsymbol{t} = a \cdot (\boldsymbol{I} - \boldsymbol{A}_{\mathcal{R}}\boldsymbol{G}_{\mathcal{R}})^{-1} \cdot \boldsymbol{t}
$$

$$
(\boldsymbol{I} - \boldsymbol{A}_{\mathcal{R}}\boldsymbol{G}_{\mathcal{R}}) \cdot \boldsymbol{t} = a \cdot \boldsymbol{t}
$$

Given us the same expression as in (4.2). ■



(a) Ring Network. *(b) Star Network.*

(c) Irregular Network . *(d) Alternate Star Network.*

Figure 2: Sample Networks from several $\boldsymbol{G}_{\mathcal{R}}$.
It is observable here that it is only fig. 2c that shows the possibility for a neutral transfer.

We further can determine the range of transfer. We then present the following results for the threshold value of neutral transfer;

Theorem 5 (Pairwise Transfer). *Assume $\{i, j\} \subsetneq \mathcal{R}$, $t_j = -t_i$ for $t_i \in \mathbb{R}_+$ and the conditions in theorem 3 holds, then the magnitude of transfer, t_i, possible such that $(x_i^t, Q_i^t) = (x_i^*, Q_i^*)$ $\forall$ agent $i \in \mathcal{R}$ is given as follows;*

$$t_i \in \left]0, \frac{n_i w_j - Q_{-j}^*}{n_i + 1}\right[. \tag{4.3}$$

Proof. WLOG, assume for $\{i, j\} \subsetneq \mathcal{R}$, $t_j = -t_i$ and $t_i \in \mathbb{R}_+$. Then from theorem 3, we have that the following equation has to hold;

$$a(w_i + t_i) + b\frac{1}{n_i}(Q_{-i}^* - t_i) < w_i + t_i. \tag{4.4}$$

Before going further, it is worth noting that we do not lose generality because take for example $\{l, m\} \subset \{\mathcal{N}_i \cap \mathcal{N}_j \cap \mathcal{R}\}$, then a transfer $t_l = -t_m$ and $t_l > 0$ means that

$$\begin{aligned} Q_{-i}^t &= \ldots + q_l^t + q_m^t + \ldots \\ &= \ldots + q_l^* + t_l + q_m^* - t_l + \ldots \\ &= Q_{-i}, \end{aligned}$$

which in turn implies $Q_{-j}^t = Q_{-j}^*$. Then from (4.4), we have that

$$\frac{1}{n_i}(Q_{-i}^* - t_i) < w_i + t_i,$$

which is the same as

$$Q_{-i}^* - n_i w_i < t_i(n_i + 1)$$

thus giving us;

$$t_i > (n_i + 1)^{-1}(Q_{-i}^* - n_i w_i). \tag{4.5}$$

Then for agent j which has $t_j = -t_i$ we intuitively multiply (4.5) by minus and replace each item accordingly so that we have;

$$t_i < (n_j + 1)^{-1}(n_j w_j - Q^*_{-j}). \tag{4.6}$$

Since from theorem 3 we know that $1 + n_i = 1 + n_j$ then $n_i = n_j$ so that we have the expression of the threshold t_i as

$$t_i \in \left] \frac{Q^*_{-i} - n_i w_i}{n_i + 1}, \frac{n_i w_j - Q^*_{-j}}{n_i + 1} \right[.$$

We also know that for any agent $i \in \mathcal{R}$, $Q^*_{-i} < n_i w_i$. This means that $Q^*_{-i} - n_i w_i < 0$ and since $t_i \in \mathbb{R}^+$ for our pairwise transfer then 0 becomes the lower bound. ■

It should be noted that $n_i = n_j$ which is an implication of theorem 3. This gives room for relatively small transfers especially as the neighbourhood size is larger. This is quite important as there are indeed utilitarian welfare implications of making transfers as discussed in the next section.

4.1 Neighbourhood Contribution Vs Individual Provision

In this part, we show that transfers from richer to poorer agents within the same neighbourhood are the most optimal transfer. We adopt the standard utilitarian welfare whereby given the Nash contribution of each agent i as q_i^*, and the sum of its neighbours $Q^*_{-i} = \sum_{j \in \mathcal{N}_i} q_j^*$, then the total welfare from the contribution game is given as $\sum_{i \in \mathcal{N}} U_i(q_i^*, Q^*_{-i}, x_i)$ Which is the sum of each agents' welfare. Going forward, we introduce another assumption which though restrictive, serves the purpose of giving us some interesting properties.

Assumption 3. *a = 0.5.*

One key motivation for this assumption as such that we now have the agent i's payoff as a function of his private consumption x_i since we now have $aw_i - 0.5q_i = 0.5x_i$. At this point, it is also observable that the payoff becomes isomorphic to that introduced in Galeotti et al. (2010) where equilibrium contributions are mapped by neighbours average contribution. We then write the following results;

Proposition 1. *Assume $t_i = 0$ for an agent $i \in \mathcal{R}$ and $\exists$ some $\{j, k\} \in \mathcal{N}_i$ such that $t_j > 0$ for which given $\boldsymbol{t}$, $\boldsymbol{x}^t = \boldsymbol{x}^*$. Then $\Delta U_i(t) = 0$ so that payoff for agent i is neutral to $\boldsymbol{t}$.*

Proof. Since we know that $q_i^t - q_i^* = t_i - (x_i^t - x_i^*)$ and $x_i^t = x_i^*$ then it means that

$$q_i^t = t_i + q_i^* = q_i^*.$$

Also, given that $Q_i^t = Q_i^*$ then,

$$Q_{-i}^t = Q_i^t - q_i^t = Q_i^* - q_i^* = Q_{-i}^*.\square$$

■

Lemma 6. *For each agent $i \in \mathcal{R}$ such that $t_i \in \mathbb{R}_+$ and $(x_i^t, Q_i^t) = (x_i^*, Q_i^*)$ we have the following welfare differential;*

$$\Delta U_i(\boldsymbol{t}) = \begin{cases} 0.5 n_i x_i^* \left(\frac{Q_i^* t_i}{Q_i^{*2} - t_i Q_{-i}^*} \right) & \textit{For } c_i = (1-a)Q_{-i} \\ 0.5 \left(\frac{n_i x_i^* Q_i^* t_i}{Q_i^{*2} - t_i Q_{-i}^*} + t_i \right) & \textit{For } c_i = 0. \end{cases} \tag{4.7}$$

Proof. If we assume $c_i = (1-a)Q_{-i}$, then we have that

$$U_i(q_i, Q_{-i}) = \frac{n_i q_i}{Q_{-i}}(a w_i - 0.5 q_i) + (1-a)Q_i. \tag{4.8}$$

assuming the transfer t_i which is *neutral* so that x_i and Q_i remain unchanged. Here, we are now focusing on agents who care/benefit from neighbourhood contributions. So if the planner makes a transfer $t_i > 0$, we then have the payoff for the agent i as;

$$U_i(\boldsymbol{t}, \mathbf{q}^t) = \frac{n_i q_i^t}{Q_{-i}^t}(a(w_i + t_i) - 0.5 q_i^t) + (1-a)Q_i^*. \tag{4.9}$$

For simplicity, we introduce the following assumption; Then the welfare differential now becomes,

$$U_i(\boldsymbol{t}, \mathbf{q}^t) - U_i(\mathbf{q}^*) = \Delta U_i(\boldsymbol{t}) = \left(\frac{q_i^t}{Q_{-i}^t} - \frac{q_i^*}{Q_{-i}^*} \right) 0.5 n_i x_i \tag{4.10}$$

Given neutrality, we know that $q_i^t + Q_{-i}^t = q_i^* + Q_{-i}^*$ which implies that

$$Q_{-i}^* = q_i^t - q_i^* + Q_{-i}^t = t_i + Q_{-i}^t.$$

Therefore we have that;

$$\begin{aligned}
\Delta U_i(\boldsymbol{t}) &= 0.5 n_i x_i^* \left(\frac{q_i^t}{Q_{-i}^t} - \frac{q_i^*}{Q_{-i}^*} \right), \\
&= 0.5 n_i x_i^* \left(\frac{q_i^t}{Q_{-i}^t} - \frac{q_i^*}{t_i + Q_{-i}^t} \right), \\
&= 0.5 n_i x_i^* \left(\frac{q_i^t \left(t_i + Q_{-i}^t\right) - Q_{-i}^t q_i^*}{Q_{-i}^t \left(t_i + Q_{-i}^t\right)} \right), \\
&= 0.5 n_i x_i^* \left(\frac{q_i^t t_i + Q_{-i}^* t_i}{Q_{-i}^t \left(t_i + Q_{-i}^t\right)} \right), \\
&= 0.5 n_i x_i^* \left(\frac{Q_i^* t_i}{Q_{-i}^t \left(t_i + Q_{-i}^t\right)} \right) = n_i x_i^* \left(\frac{Q_i^* t_i}{Q_{-i}^t \left(Q_i^* - q_i^*\right)} \right), \\
&= 0.5 n_i x_i^* \left(\frac{Q_i^* t_i}{Q_{-i}^t Q_{-i}^*} \right) = n_i x_i^* \left(\frac{Q_i^* t_i}{\left(Q_i^* - \left(t_i + q_i^*\right)\right)\left(Q_i^* - q_i^*\right)} \right), \\
&= 0.5 n_i x_i^* \left(\frac{Q_i^* t_i}{Q_i^{*2} - t_i Q_{-i}^*} \right)
\end{aligned}$$

On the other hand where $c_i = 0$ we have $(1-a)q_i^t - (1-a)q_i^* = (1-a)t_i$ so that we now have

$$\Delta U_i(\boldsymbol{t}) = 0.5 n_i x_i^* \left(\frac{Q_i^* t_i}{Q_i^{*2} - t_i Q_{-i}^*} \right) - 0.5 t_i.$$

■

The information above provides us with the intuition that since $|t_i| < Q_{-i} < Q_i$, then if $t_i > 0$, $\Delta U_i(\boldsymbol{t}) > 0$ and vice versa. However, for the broad welfare from this model, we use pairwise equilibrium to generalise the conditions for the welfare improvement. This is given the theorem 5 proof where we see that welfare of neighbours other than those whom the transfer is between is neutral to such transfer. We then have the following results;

Theorem 6. *Assuming a pairwise transfer $t_i > 0$, welfare differential $\sum_{i \in \mathcal{R}} \Delta U_i(\boldsymbol{t})$ is invariant of c_i for all $i \in \mathcal{R}$.*

Proof. Here we are able to hold $a \in]0,1[$ as being unspecified. We already know that given $t_i \neq 0$ for an agent $i \in \mathcal{R}$, where $c_i = (1-a)Q_{-i}$, then given $U_i(q_i, Q_{-i}) = a(\cdot)_i + (1-a)Q_i$, then $(1-a)Q_i^t - (1-a)Q_i^* = 0$.

However, say $c_i \neq (1-a)Q_{-i}$ then we have $\Delta U_i(\boldsymbol{t}) = a(\cdot)_i + (1-a)t_i$ since in that case $t_i : t_i \to c_i(t_i) = 0$. But then we recall that $\exists$ an agent j such that $t_j = -t_i$ so that for j, we have $\Delta U_j(\boldsymbol{t}) = a(\cdot)_j - (1-a)t_i$ which still means that

$$\sum_{i \in \mathcal{R}} \Delta U_i(\boldsymbol{t}) = a(\cdot)_i + a(\cdot)_j.$$

■

This is quite interesting as we see here that while lemma 6 shows that the agent i who receives t_i potentially experiences a welfare improvement which is only increased if his payoff is dependent on neighbourhood provision, the exact improvement is offset by the loss in the welfare of the agent j whose $t_j = -t_i$ such that aggregate payoff evens out. One question that might potentially arise from this result is why we consider $c_i > 0$ to begin with. The simple reason is that with a non-neutral transfer, $c_i(Q_{-i}) = (1-a)Q_{-i}$ for example means that $c_i + (1-a)q_i^* = (1-a)Q_i$ is such that $(1-a)\left(Q_i^t - Q_i^*\right) \neq 0$ which then affects overall welfare differently. Also, we specify further conditions for welfare improvement as follows;

Theorem 7. *For every pair nodes, say agents $\{i,j\} \subset \mathcal{R}$ such that $t_j = -t_i$ for $t_i \in \mathbb{R}_+$ and $\boldsymbol{t}$ is a neutral transfer vector, there exists a non-Pareto welfare improvement i.e, $\sum_{i \in \mathcal{R}} \Delta U_i(\boldsymbol{t}) \in \mathbb{R}_{++}$ when then the following condition holds;*

$$x_j^* - x_i^* < \frac{t_i}{Q_i^*}\left(x_i^* + x_j^*\right). \tag{4.11}$$

Proof. WLOG, Assume $\mathcal{R} = \{i, j, \ldots\}$ such that agent $j \in \mathcal{N}_i$ and $\{i,j\} \in \mathcal{N}_{-i \neq j}$. Then assume $t_i + t_j = 0$ such that $t_i > 0$. we have;

$$\sum_{i\in\mathcal{R}} \Delta U_i(\boldsymbol{t}) = \frac{0.5\left(n_i x_i^* Q_i^*\left(Q_j^{*2} + t_i Q_{-j}^*\right)t_i - n_j x_j^* Q_j^*\left(Q_i^{*2} + t_i Q_{-i}^*\right)\right)t_i}{\left(Q_i^{*2} - t_i Q_{-i}^*\right)\left(Q_j^{*2} + t_i Q_{-j}^*\right)},$$

$$= \frac{0.5\left(n_i x_i^* Q_i^* Q_j^{*2} + t_i n_i x_i^* Q_i^* Q_{-j}^* - n_j x_j^* Q_j^* Q_i^{*2} + t_i n_j x_j^* Q_j^* Q_{-i}^*\right)t_i}{\left(Q_i^{*2} - t_i Q_{-i}^*\right)\left(Q_j^{*2} + t_i Q_{-j}^*\right)},$$

$$= \frac{0.5\left(n_i x_i^* Q_i^* Q_j^{*2} - n_j x_j^* Q_j^* Q_i^{*2} + t_i Q_i^* Q_{-j}^*(n_i x_i^* + n_j x_j^*)\right)t_i}{\left(Q_i^{*2} - t_i Q_{-i}^*\right)\left(Q_j^{*2} + t_i Q_{-j}^*\right)},$$

$$= \frac{0.5\left(n_i x_i^* Q_j^* - n_j x_j^* Q_i^* + t_i(n_i x_i^* + n_j x_j^*)\right)Q_i^* Q_j^* t_i}{\left(Q_i^{*2} - t_i Q_{-i}^*\right)\left(Q_j^{*2} + t_i Q_{-j}^*\right)},$$

$$= \frac{0.5\left(n_i x_i^*(Q_j^* + t_i) + n_j x_j^*(t_i - Q_i^*)\right)Q_i^* Q_j^* t_i}{\left(Q_i^{*2} - t_i Q_{-i}^*\right)\left(Q_j^{*2} + t_i Q_{-j}^*\right)},$$

since we know that $t_i < n_i^{-1} Q_{-i}$ then we can claim that $Q_i^{*2} - t_i Q_{-i}^* > 0$ for any t_i within the threshold in theorem 5. The computation above then implies that for $\sum_{i\in\mathcal{R}} \Delta U_i(\boldsymbol{t}) \in \mathbb{R}_+$, it has to hold that $n_i x_i^*(Q_j^* + t_i) + n_j x_j^*(t_i - Q_i^*) > 0$ in which such a condition is rewritten as $n_i x_j^* Q_i^* < n_i x_i^*(Q_j^* + t_i) + n_j x_j^* t_i$. But we know from theorem 3 that $n_i = n_j$ and $Q_i^* = Q_j^*$. So then when we multiply by n_i^{-1}, we have

$$x_j^* Q_i^* - x_i^* Q_i^* < x_i^* t_i + x_j^* t_i.$$

■

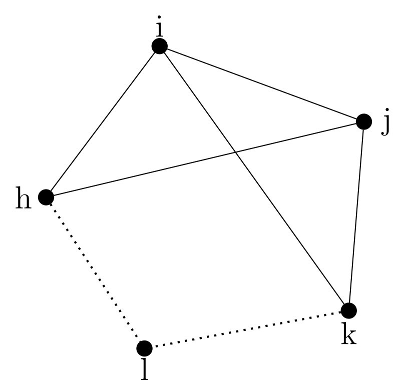


Figure 3: Network with one poor agent

Example 2. Assume a network with $\mathcal{N} = \{h, i, j, k, l\}$ as in fig. 3 where agent $l \in \mathcal{N}/\mathcal{R}$ and $a = 0.5$. We can see that a neutral transfer $t_i > 0$ for $t_i + t_j = 0$ is feasible if within the threshold as in eq. (4.3). Assuming $w_i = w_j$, then we know that $(x_i^*, Q_i^*) = (x_j^*, Q_j^*)$. This means that it is impossible to meet the threshold $t_i \in \left]0, \frac{3w_i - Q_{-i}^*}{4}\right[$ since $3w_i - Q_{-i}^* = -(Q_{-i}^* - 3w_i)$ but $Q_{-i}^* - 3w_i > 0$.

Observing the proposition above, if we assume that $x_i^* < x_j^*$ then $x_i^* - x_j^* > 0$ such that transfer t_i need be large enough to bridge the ratio gap otherwise the planner faces a welfare loss due to such transfer policies. Another possible implication of the result above is that neutral pairwise transfer is impossible among 2 agents with identical neighbourhoods who in turn possess identical wealth endowment.

5 Conclusions

In this work, we introduce and identify some early properties of collective action problems where the decisions of agents are dependent on the extent of contributions of neighbours. Such interactions always lead to at least one corner contributing agent such that the extent of interior contributions becomes increasingly dependent on the weighted nearness to such corner agent. We also see how neutral transfers are capable of improving overall welfare in such a network. We observe how this improvement holds overall even in cases where neighbourhood provision is of value to a representative agent.

It should be noted that our emphasis on pairwise transfer does not limit our results especially given that such transfers are neutral. Neutral transfers rely on strict conditions such that they are only achieved if such transfer within the neighbourhood is likened to a simple pairwise transfer between 2 agents. A noticeable skepticism may arise through the concept of attribution peer effect to mean-neighbours contributions. This may be due to alternative forms including instances where peer-based contributions are a direct function of the next largest contribution so that agents are only best off should they be the largest neighbourhood contributor. This may be worth exploring. Also is the question of optimal budget-balanced transfer which may not be transfer-neutral and welfare properties of such transfer.

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